Scholars Journal of Engineering and Technology (SJET) **ISSN 2321-435X (Online)**
Sch. J. Eng. Tech., 2014; 2(5A):693-698 **ISSN 2347-9523 (Print)**

(An International Publisher for Academic and Scientific Resources)
www.saspublisher.com

Research Article

Existence of Positive Solutions for Nonlinear Fourth-order Boundary Value Problems

Jing Ge
School of Mathematical and Statistics, Northeast Petroleum University, Daqing, Heilongjiang, 163318, PR China

***Corresponding author**
Jing Ge
Email: gesaki@163.com

Abstract: The nonlinear fourth-order boundary value problem

$$u^{(4)}(t)-\rho^4 u(t)=f\left(t,u(t)\right)$$
$$u(1)=u'(0)=u''(1)=u'''(0)=0$$

is studied in this paper, where $0<\rho<\frac{\pi}{2}$ .The existence result of at least one positive solution to above fourth-order boundary value problem is obtained by using Fixed Point theorem in cones.
Keywords: Fourth-order boundary value problem, Cone, Positive solutions, Fixed point
MSC: 34B10, 34B15

INTRODUCTION

In this paper, we consider the nonlinear fourth-order boundary value problems (BVP for short)

$$u^{(4)}(t)-\rho^4 u(t)=f\left(t,u(t)\right),0<t<1, \tag{1.1}$$

where $0<\rho<\frac{\pi}{2}$ is a parameter , $f:[0,1]\times[0,+\infty)\to R$ is a nonnegative and continuous function. With boundary conditions

$$u(1)=u'(0)=u''(1)=u'''(0)=0 \tag{1.2}$$

In this case, a beam deformation with one endpoint simply supported and the other one sliding clamped. By a positive solution of BVP (1.1) and (1.2) ,we call a function $u(t)$ which is positive on $(0,1)$ and $u(t)\in C^3[0,1]\cap C^4[0,1]$ such that $u(t)$ satisfied differential equation (1.1) and the boundary conditions (1.2) .It is assumed throughout that

(H_1): $f(t,u)$ is integral for each fixed $u\in[0,1]\times[0,+\infty)$, and $0<\int_0^1 f\left(t,u(t)\right)\mathrm{d}t<+\infty$;

(H_2): $\lim\limits_{u\to 0^+}\sup\limits_{t\in[0,1]}\frac{f(t,u)}{u}=0, \lim\limits_{u\to\infty}\inf\limits_{t\in[0,1]}\frac{f(t,u)}{u}=\infty$; (H_3): $\lim\limits_{u\to 0^+}\sup\limits_{t\in[0,1]}\frac{f(t,u)}{u}=\infty, \lim\limits_{u\to\infty}\inf\limits_{t\in[0,1]}\frac{f(t,u)}{u}=0.$

The nonlinear fourth-order equations appear in some physical problems as, for example, the bending of an elastic beam with several types of two point boundary conditions, describing how the beam is supported at the two endpoints, see [1-13] .The positive solution has profound practical significance.

In 1996, Dalmasso first proved the existence of single positive solution of problem

$$u^{(4)}(t)=h(t)f\left(t,u(t),u'(t)\right),t\in[0,1]\setminus E$$

under two -point boundary conditions

$$u(0)=u'(0)=u''(1)=u'''(1)=0$$

By comparing the first value of associated linear problem with the limits

$$\limsup_{u\to 0}\frac{f(u)}{u}\quad \liminf_{u\to\infty}\frac{f(u)}{u}$$

When $E\neq\phi, h(t)\equiv 1, f(t,u)=f(u)$ and $f:[0,+\infty]\to[0,+\infty]$ is continuous.

For boundary conditions considering every derivative until order there, similar results can be obtained, since the second and third derivative are given in different endpoints. More precisely, considering Eq.(1.1) with one of the following boundary conditions

$$u(0)=u'(1)=u''(1)=u'''(0)=0$$

$$u(1)=u'(0)=u''(0)=u'''(1)=0$$

Inspired and motivated by the works mentioned above, in this paper, we will consider the existence of positive solution to the nonlinear BVP (1.1) and (1.2). The purpose of this paper is to fill in the gap in this area.The results obtained extend and complement some known results.

The rest of the article is organized as follows, In Section 2, we present some preliminaries and the fixed point theory in cone that willed be used in Section 3.The main results and proofs will be given in Section 3.

PRELIMINARIES AND LEMMAS

Consider the Banach space $C[0,1]$ with norm $\|u\|=\max_{0\le t\le 1}|u(t)|$ and let

$$C^+[0,1]=\{u\in C[0,1]; u(t)\ge 0, 0\le t\le 1\},\ K=\left\{u(t)\in C^+[0,1];\ \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]}u(t)\ge\sigma\|u\|, 0\le t\le 1\right\},$$

$$\sigma=\frac{3\rho}{16}\operatorname{csch}(\rho)$$

it is easy to check that K is a cone of nonnegative function in $C[0,1]$.

Consider the nonlinear second order boundary problem

$$u''(t)-\rho^2u(t)=-v(t) \tag{2.1}$$

$$u(1)=0, u'(0)=0 \tag{2.2}$$

A direct check implies that the problem (2.1), (2.2) is equivalent to the following integral equation

$$u(t)=\int_0^1 G_1(t,s)v(s)\mathrm{d}s, \tag{2.3}$$

Where

$$G_1(t,s)=\begin{cases}\dfrac{\cosh(\rho s)\sinh(\rho-\rho t)}{\rho\cosh(\rho)}, 0\le s\le t\le 1\\[2ex] \dfrac{\cosh(\rho t)\sinh(\rho-\rho s)}{\rho\cosh(\rho)}, 0\le t\le s\le 1\end{cases} \tag{2.4}$$

Consider the nonlinear second order boundary problem

$$v''(t)+\rho^2v(t)=-f(t,u(t)) \tag{2.5}$$

$$v(1)=0, v'(0)=0 \tag{2.6}$$

The same method

$$v(t)=\int_0^1 G_2(t,s)f(s,u(s))\mathrm{ds} \tag{2.7}$$

where

$$G_2(t,s)=\begin{cases}\dfrac{\sin\rho(1-t)\cos\rho s}{\rho\cos\rho},0\le s\le t\le 1\\ \dfrac{\sin\rho(1-s)\cos\rho t}{\rho\cos\rho},0\le t\le s\le 1\end{cases} \qquad (2.8)$$

we can easily compute $u(t)=\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\mathrm{d}s$

Clearly if $u(t)$ is a positive solution of the problem (2.1) and (2.2) and let $u(t)=\Phi u(t)$, it is easy to know $u(t)$ is the positive solution of the BVP (1.1), (1.2).

Lemma 2.1: Foe all $(s,t)\in[0,1]\times[0,1]$, we have

$$\frac{G_1(t,s)}{G_1(s,s)}=\begin{cases}\dfrac{\sinh(\rho-\rho t)}{\sinh(\rho-\rho s)},0\le s\le t\le 1\\ \dfrac{\cosh(\rho t)}{\cosh(\rho s)},0\le t\le s\le 1\end{cases}$$

$$\rho t(1-t)\operatorname{csch}(\rho)G_1(s,s)\le G_1(t,s)\le G_1(s,s)$$

Proof: It is clearly to see

$$\frac{G_1(t,s)}{G_1(s,s)}=\begin{cases}\dfrac{\sinh(\rho-\rho t)}{\sinh(\rho-\rho s)},0\le s\le t\le 1\\ \dfrac{\cosh(\rho t)}{\cosh(\rho s)},0\le t\le s\le 1\end{cases}$$

$$\ge\begin{cases}\rho t(1-t)\operatorname{csch}(\rho),0\le s\le t\le 1\\ \rho t\operatorname{sech}(\rho),0\le t\le s\le 1\end{cases}$$

$$\ge \rho t(1-t)\operatorname{csch}(\rho)$$

It is obvious that $G_1(t,s)\le G_1(s,s)$.The proof is complete.

Define an integral operator $\Phi: C^+[0,1]\to C^+[0,1]$ by

$$\Phi u(t)=\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\mathrm{d}s \qquad (2.9)$$

Then, only if nonzero fixed point $u(t)$ of mapping Φ defined by (2.9) is a positive solution of (1.1) and (1.2)

Lemma 2.2: $\Phi(K)\subset K$

Proof: For any $u\in K$, from lemma 2.1we have

$$\|\Phi u(t)\|=\max\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\,\mathrm{d}s$$

And inequalities

$$\|\Phi u(t)\|\le\int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\,\mathrm{d}s$$

$$\min_{t\in\left[\frac14,\frac34\right]}\Phi u(t)=\min_{t\in\left[\frac14,\frac34\right]}\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\,\mathrm{d}s$$

$$\ge\rho t(1-t)\operatorname{csch}(\rho)\int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)f(\tau,u(\tau))\mathrm{d}\tau\,\mathrm{d}s$$

$$\geq \sigma \int_0^1 \int_0^1 G_1(s,s) G_2(s,\tau) f(\tau, u(\tau)) \mathrm{d}\tau \mathrm{d}s$$

$$\geq \sigma \|\Phi u(t)\| = \sigma \|\Phi u\|$$

Thus, $\Phi(k) \subset K$

It is clear that $\Phi : K \to K$ is a completely continuous mapping.

Let E be a Banach space, and let $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$ and let $\Phi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either

(1) $\| \Phi u \| \leq \| u \|$, $u \in K \cap \partial\Omega_1$, and $\| \Phi u \| \geq \| u \|$, $u \in K \cap \partial\Omega_2$; or

(2) $\| \Phi u \| \geq \| u \|$, $u \in K \cap \partial\Omega_1$, and $\| \Phi u \| \leq \| u \|$, $u \in K \cap \partial\Omega_2$,

Then Φ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

We will apply the first and second parts of the above Fixed Point Theorem to the super-linear and sub-linear cases.

MAIN RESULTS

Theorem 3.1: Assume that $(H_1), (H_2)$ hold, then the problem (1.1) and (1.2) has at least one positive solution.

Proof: Since (H_2), we may choose $r > 0$ so that $f(t,u) \leq \varepsilon u$, for $0 \leq u \leq r$, where $\varepsilon > 0$ satisfies

$$\varepsilon \int_0^1 \int_0^1 G_1(s,s) G_2(s,\tau) f(\tau, u(\tau)) \mathrm{d}\tau \mathrm{d}s \leq 1.$$

Let, $\Omega_1 = \{u \in C[0,1]; \| u \| < r\}$, $\forall u \in K \cap \partial\Omega_1$ from lemma 2.1, we have

$$\|\Phi u(t)\| \leq \int_0^1 \int_0^1 G_1(s,s) G_2(s,\tau) f(\tau, u(\tau)) \mathrm{d}\tau \mathrm{d}s$$

$$\leq \varepsilon \|u\| \int_0^1 \int_0^1 G_1(s,s) G_2(s,\tau) \mathrm{d}\tau \mathrm{d}s$$

$$\leq \|u\|$$

Then shows $\| \Phi u \| \leq \| u \|$.

Further, since (H_2) there exists $R_1 > 0$ such that $f(t,u) \geq \mu u$, $u \geq R_1$ where $\mu > 0$ chosen so that

$$\mu\sigma \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,s) G_2(s,\tau) \mathrm{d}\tau \mathrm{d}s \geq 1$$

Let $R > \max\{r, \frac{R_1}{\sigma}\}$ and $\Omega_2 = \{u \in C[0,1]; \| u \| < R\}$, then $\forall u \in K \cap \partial\Omega_2$ and

$\min\limits_{t \in [1/4, 3/4]} u(t) \geq \sigma \| u \| = \sigma R > R_1$, implies

$$\|\Phi u(t)\| \geq \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} \sigma \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,s) G_2(s,\tau) f(\tau, u(\tau)) \mathrm{d}\tau \mathrm{d}s \quad \geq \sigma\mu \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,s) G_2(s,\tau) u(\tau) \mathrm{d}\tau \mathrm{d}s$$

$$\geq \|u\|$$

Hence, $\| \Phi u \| \geq \| u \|$ for $\forall u \in K \cap \partial\Omega_2$

Therefore, by the first part of the Fixed Point Theorem, it follows that Φ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. Further, since $G_1(t,s) G_2(s,\tau) \mathrm{d}\tau \mathrm{d}s \geq 0$, it follows that $u(t) > 0$ for $0 < t < 1$. This completes the super-linear part of the theorem.

Theorem 3.2: Assume that $(H_1), (H_3)$ hold, then the Problem (1.1) and (1.2) has at least one positive solution.

Proof: Since (H_3) we first choose $r > 0$ such that $f(t,u) \geq \mu u$, for $0 \leq u \leq r$ where $\mu > 0$ satisfies

$$\mu\sigma\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}G_1(s,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s\geq 1,$$

Let $\Omega_1=\{u\in C[0,1];\|u\|<r\}$, for $\forall u\in K\bigcap\partial\Omega_1$, from lemma 2.1, we have

$$\begin{aligned}\left\|\Phi u(t)\right\|&\geq \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}G_1(s,s)G_2(s,\tau)f\left(\tau,u(\tau)\right)\mathrm{d}\tau\mathrm{d}s\\ &\geq\sigma\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}G_1(s,s)G_2(s,\tau)f\left(\tau,u(\tau)\right)\mathrm{d}\tau\mathrm{d}s\\ &\geq\mu\sigma\|u\|\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}G_1(s,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s\geq\|u\|\end{aligned}$$

So that $\|\Phi u\|\geq\|u\|$.

Now since (H_3), there exists $R_1>0$ so that $f(t,u)\leq\varepsilon u$, for $u\geq R_1$ where $\varepsilon>0$ satisfies $\varepsilon\int_0^1\int_0^1G_1(s,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s<1$. $\forall u\in K\bigcap\partial\Omega_1$.

We consider two case:

Suppose $f(t,u)$ is unbounded for $\forall 0<u\leq R$, we have $f(u)\leq f(R)$, $R>\max\{r,\frac{R_1}{\sigma}\}$, $\min\limits_{t\in[1/4,3/4]}u(t)\geq\sigma\|u\|=\sigma R>R_1$.

Let $\Omega_2=\{u\in C[0,1];\|u\|<R\}$, for $\forall u\in K\bigcap\partial\Omega_2$ therefore

$$\begin{aligned}\left\|\Phi u(t)\right\|&\leq\int_0^1\int_0^1G_1(s,s)G_2(s,\tau)f\left(\tau,u(\tau)\right)\mathrm{d}\tau\mathrm{d}s\\ &\leq\int_0^1\int_0^1G_1(s,s)G_2(s,\tau)f\left(R\right)\mathrm{d}\tau\mathrm{d}s\\ &\leq\varepsilon R\int_0^1\int_0^1G_1(s,s)G_2(s,\tau)R\mathrm{d}\tau\mathrm{d}s\leq R=\|u\|\end{aligned}$$

So that $\|\Phi u\|\leq\|u\|$.

Suppose $f(t,u)$ is bounded, there exists $N>0$, for $t\in[0,1]$ and $u\in[0,+\infty)$ we have $f(t,u)\leq N$, $R>\max\{r,N\int_0^1\int_0^1G_1(s,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s\}$,

Let $\Omega_2=\{u\in C[0,1];\|u\|<R\}$, for $\forall u\in K\bigcap\partial\Omega_2$, from lemma 2.1, we have

$$\begin{aligned}\left\|\Phi u(t)\right\|&\leq\int_0^1\int_0^1G_1(s,s)G_2(s,\tau)f\left(\tau,u(\tau)\right)\mathrm{d}\tau\mathrm{d}s\\ &\leq N\int_0^1\int_0^1G_1(s,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s\\ &\leq R=\|u\|\end{aligned}$$

So that $\|\Phi u\|\leq\|u\|$.

Therefore, in either case we may put $\Omega_2=\{u\in C[0,1];\|u\|<R\}$ and for $\forall u\in K\bigcap\partial\Omega_2$ we have $\|\Phi u\|\leq\|u\|$. By the second part of the Fixed Point Theorem it follows that BVP $(1.1),(1.2)$ has a positive solution, and this completes the proof of the theorem.

REFERENCES

1. Gupta CP; Existence and uniqueness theorem for a bending of an elastic beam equation. Appl Anal., 1988; 26(4): 289–304.
2. Agarwal RP; On fourth-order boundary value problems arising in beam analysis. Differential Integral Equations, 1989; 2(1): 91–110.
3. Gupta CP; Existence and uniqueness results for a bending of an elastic beam equation at resonance. J Math Anal

Appl., 1998; 135(1): 208–225.
4. Gupta CP; Existence and uniqueness results for some fourth order fully quasilinear boundary value problems. Appl Anal.,1990; 36: 169–175.
5. Del Pino MA, Manasevich RF; Existence for a fourth-order boundary value problem under a two-parameter non resonance condition. Proc Amer Math Soc., 1991; 112(1): 81–86.
6. Aftabizadeh AR; Existence and uniqueness theorems for fourth-order boundary value problems. J Math Anal Appl., 1986; 116(2): 415–426.
7. Ma RY, Wang H; On the existence of positive solutions of fourth-order ordinary differential equations. Appl Anal., 1995; 59: 225–231.
8. Yang Y; Fourth-order two-point boundary value problem. Proc. Amer Math Soc., 1988; 104(1): 175–180.
9. De Coster C, Fabry C, Munyamarere F; Non resonance conditions for fourth order nonlinear boundary value problems. Internat J Math Sci., 1994; 17: 725–740.
10. Agarwal RP, O'Regan D; Existence theorem for single and multiple solutions to singular position boundary value problems. J Differential Equations, 2001; 175: 393–414.
11. Gupta CP; Existence and uniqueness theorems for a fourth order boundary value problem of Sturm-Liouville type. Differential Integral Equations, 1991; 4(2): 397-410.
12. Gyulov T, Tersian S; Existence of trivial and nontrivial solutions of a fourth ordinary differential equation. Electron J Diff Eqns., 2004; 41: 1-14.
13. Ma TF, da Silva J; Iterative Solutions for a beam equation with nonlinear boundary conditions of third order. Appl Math Comput., 2004; 159(1): 11-18.