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## Research Article

# On Homogeneous Ternary Quadratic Diophantine Equation $4(x^2+y^2)-7xy=16\,z^2$

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**Abstract:** The ternary quadratic homogeneous equation representing homogeneous cone given by $4(x^2+y^2)-7xy=16\,z^2$ is analyzed for its non-zero distinct integer points on it. Five different patterns of integer points satisfying the cone under consideration are obtained. A few interesting properties among the solutions and polygonal numbers are presented.

**Keywords:** Ternary homogeneous quadratic, Integral solutions.

**INTRODUCTION:**

The ternary quadratic Diophantine equations offer an unlimited field for research due to their variety [1, 21]. For an extensive review of various problems, one may refer [2-20]. This communication concerns with yet another interesting ternary quadratic equation $4(x^2+y^2)-7xy=16\,z^2$ representing a cone for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

**Notation:** $t_{m,n}$ - Polygonal number of rank n with size m.

**METHOD OF ANALYSIS**

The ternary quadratic equation to be solved for its non-zero distinct integer solution is

$$4(x^2+y^2)-7xy=16\,z^2 \tag{1}$$

Note that (1) is satisfied by (8, 6, 2), (2, 0, 1), $(2k^2+4k-6, 2k^2-8, t_{3,k}+2)$ and $(30k^2+32k+8, 30k^2+28k+6, 15t_{3,k}+2)$. However, we have the other choices of solutions which are illustrated below. The substitution of the linear transformations

$$x = U + V,\ y = U - V \tag{2}$$

in (1) leads to

$$U^2 = 16z^2 - 15V^2 \tag{3}$$

**Pattern: 1**

$$\text{Consider } z = X - 15T,\ V = X - 16T \tag{4}$$

The substitution of (4) in (3) leads to

$$U^2 + 240\,T^2 = X^2 \tag{5}$$

One may write (5) as

$$U^2 + 15\,(4T)^2 = X^2 = X^2 * 1 \tag{6}$$

$$\text{Assume } X = X(a, b) = a^2 + 15\,b^2 \tag{7}$$

$$\text{Write 1 as , } 1 = \frac{(1+i\sqrt{15})(1-i\sqrt{15})}{16} \tag{8}$$

Substituting (7), (8) in (6) and employing the method of factorization, define

$$\left(U + i\sqrt{15}(4T)\right) = \left(a + i\sqrt{15}b\right)^2 \frac{(1 + i\sqrt{15})}{4}$$

Equating real and imaginary parts, we have .

$$\left.\begin{aligned} U &= \frac{1}{4}\,[\,a^2 - 30ab - 15b^2] \\ T &= \frac{1}{16}\,[\,a^2 + 2ab - 15b^2] \end{aligned}\right\} \quad (9)$$

The choices a = 4A, b = 4B in (9) and (7) give

$$\left.\begin{aligned} U &= U\,(A,B) = 4A^2 - 120AB - 60B^2 \\ T &= T\,(A,B) = A^2 + 2AB - 15B^2 \\ X &= X(A,B) = 16A^2 + 240B^2 \end{aligned}\right\} \quad (10)$$

From (10), (4) and (2), the integer values of x, y and z satisfying (1) are given by

$$x = x(A,B) = 4A^2 - 152AB + 420B^2$$
$$y = y(A,B) = 4A^2 - 88AB - 540B^2$$
$$z = z(A,B) = A^2 - 30AB + 465B^2$$

**Properties**

- $x(A, 1) - t_{308,A} + t_{297,A} \equiv -24 \pmod{48}$
- $x(1, B) + y(1, B) + z(1, B) - t_{692,B} \equiv 9 \pmod{74}$
- $y(A, 1) - t_{180, A} + t_{172, A} \equiv 48 \pmod{84}$
- $y(A, 2) + z(A, 2) - t_{12, A} \equiv 164 \pmod{232}$
- $x(1, A) - t_{842, A} \equiv 4 \pmod{267}$
- $x(1, A) - z(1, A) + t_{92, A} \equiv 3 \pmod{166}$

**Pattern: 2**

Instead of (4) consider

$$z = X + 15T,\ V = X + 16T \quad (11)$$

The substitution of (11) in (3) leads to

$$X^2 = 240\,T^2 + U^2$$

which is satisfied by

$$\left.\begin{aligned} T\,(A,B) &= 2AB \\ U\,(A,B) &= 240A^2 - B^2 \\ X\,(A,B) &= 240A^2 + B^2 \end{aligned}\right\} \quad (12)$$

From (11), (12) and (2), the integer values of x, y and z satisfying (1) are given by

$$x = x(A,B) = 480A^2 + 32AB$$
$$y = y(A,B) = -32AB - 2B^2$$
$$z = z(A,B) = 240A^2 + 30AB + B^2$$

**Properties**

- $x(1, B) \equiv 0 \pmod{32}$
- $x(1, B) + y(1, B) + z(1, B) + t_{4,B} \equiv 0 \pmod{30}$
- $y(2, B) + z(2, B) + t_{4,B} \equiv 0 \pmod{4}$
- $y(2, B) - z(2, B) + t_{8,B} \equiv 48 \pmod{126}$

**Pattern: 3**

Introduction of the linear transformation

$$U = X - 15T, V = X + T \quad (13)$$

in (3) leads to

$$X^2 + 15T^2 = Z^2 = Z^2 * 1 \quad (14)$$

Assume $z = a^2 + 15b^2$ (15)

Write 1 as $1 = \frac{(1+i\sqrt{15})(1-i\sqrt{15}}{16}$ (16)

Substituting (15), (16) in (14) and employing the method of factorization, define

$$\left(X + i\sqrt{15}\,T\right) = \left(a + i\sqrt{15}b\right)^2 \frac{(1 + i\sqrt{15})}{4}$$

Equating real and imaginary parts, we have

$$\left.\begin{aligned} X &= \frac{1}{4}\,[\,a^2 - 30ab - 15b^2] \\ T &= \frac{1}{4}\,[\,a^2 - 2ab - 15b^2] \end{aligned}\right\} \quad (17)$$

The choices a = 4A, b = 4B in (15) and (17) lead to

$$\left.\begin{aligned} X &= 4A^2 - 120AB - 60B^2 \\ T &= 4A^2 + 8AB - 60B^2 \\ z &= 16A^2 + 240B^2 \end{aligned}\right\} \quad (18)$$

From (17), (18) and (13), the integer values of x, y and z satisfying (1) are given by

$$x = x(A, B) = -48A^2 - 352AB + 720B^2$$
$$y = y(A, B) = -64A^2 - 128AB + 960B^2$$
$$z = z(A, B) = 16A^2 + 240B^2$$

**Properties**

- $x(A, 1) - y(A, 1) - t_{34, A} \equiv 178 \pmod{209}$
- $x(A, A+1) - t_{642, A} \equiv 720 \pmod{1407}$
- $x(1, B) - y(1, B) + z(1, B) \equiv 32 \pmod{224}$
- $x(2, A) + z(2, A) - 960t_{4, A} \equiv -128 \pmod{704}$

**Pattern: 4**

Instead of (13), consider

$$U = X + 15T, V = X - T \quad (19)$$

The substitution of (19) in (3) leads to

$z^2 = X^2 + 15T^2$

which is satisfied by

$$\left.\begin{aligned} T\,(A, B) &= 2AB \\ X\,(A, B) &= 15A^2 - B^2 \\ Z\,(A, B) &= 15A^2 + B^2 \end{aligned}\right\} \quad (20)$$

From (20), (19) and (2), the integer values of x, y and z satisfying (1) are given by

$$x = x(A, B) = 30A^2 + 28AB - 2B^2$$

$y = y(A,B) = 32AB$
$z = z(A,B) = 15A^2 + B^2$

**Properties**

- $x(A, 1) - y(A, 1) - t_{62,A} \equiv -2 \pmod{25}$
- $x(1, A) - y(1, A) - z(1, A) + 6\, t_{3,A} + A = 15$
- $x(A, A+1) - 28\, t_{4,A} - 56\, t_{3,A} \equiv -2 \pmod 4$

**PATTERN: 5**

Write (3) as

$$U^2 - z^2 = 15\, z^2 - 15V^2 \qquad (21)$$

Factorizing (21) we have

$$(U + z)(U - z) = 15\, (z + V)(z - V)$$

which is equivalent to the system of double equations

$$BU - AV + (B - A)\, z = 0$$
$$- AU - 15\, BV + (15B + A)\, z = 0, \text{ where } A, B \neq 0$$

Applying the method of cross multiplication, we get

$U = -A^2 - 30AB + 15B^2$
$V = A^2 - 2AB - 15B^2$
$z = -A^2 - 15B^2$

Employing (2) the values of x, y and z satisfying (1) are given by

$x = x(A,B) = -32AB$
$y = y(A,B) = -2A^2 - 28AB + 30B^2$
$z = z(A,B) = -A^2 - 15B^2$

**Properties**

- $x(1, B) - y(1, B) + t_{62,B} \equiv 2 \pmod{33}$
- $x(A, 2) + y(A, 2) + z(A, 2) + 6t_{3,A} \equiv 60 \pmod{117}$
- $x(A, A+1) + z(A, A+1) + 64\, t_{3,A} + t_{34,A} \equiv 0 \pmod{15}$

**Remarkable Observations**

Let p, q be any two non-zero distinct positive integers such that $p > q > 0$.

Define $p = x + y$ and $q = y$. Treat p, q as the generators of the Pythagorean triangle $T(\alpha, \beta, \gamma)$ where $\alpha = 2pq$, $\beta = p^2 - q^2$, $\gamma = p^2 + q^2$. Let P, A represent the perimeter and the area of T. Then each of the following expressions is a perfect square.

a: $6\gamma - 4\alpha - 2\beta - \frac{7}{2}\sqrt{2(\gamma - \alpha)(\gamma - \beta)}$

b: $4\gamma - 2\alpha - \frac{8A}{P} - \frac{7}{2}\sqrt{2(\gamma - \alpha)(\alpha - \frac{4A}{P})}$

c: $8\gamma - 6\alpha - 4\beta + \frac{8A}{P} - \frac{7}{2}\sqrt{2(\gamma - \alpha)(\alpha - \frac{4A}{P})}$

**CONCLUSION**

In this paper, we have presented five different patterns of non-zero distinct integer solutions of the homogeneous cone given by $4(x^2 + y^2) - 7xy = 16\, z^2$. To conclude, one may search for other patterns of non-zero integer distinct solutions and their corresponding properties.

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