Research Article

On sandwich theorems for certain analytic functions

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Abstract: In this paper, we derive some subordination and superordination results for certain normalized analytic functions in the open unit disk.

Keywords: univalent functions; starlike functions; subordination; superordination

INTRODUCTION

Let $H$ denote the class of analytic functions in the open unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$, and $H[a,n]$ denote the subclass of the functions $f \in H$ of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots. \quad (1)$$

Also, let $A$ be the subclass the functions $f \in H$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots. \quad (2)$$

A function $f \in A$ is said to be in the class $S^*$ of starlike functions in $U$, if it satisfies the inequality $Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in U$. Furthermore, a function $f \in A$ is said to be in the class $C$ of convex functions in $U$, if it satisfies the inequality $Re(1 + \frac{zf''(z)}{f'(z)}) > 0, z \in U$.

Let $f(z)$ and $F(z)$ be analytic in $U$, then we say that the function $f(z)$ is subordinate to $F(z)$ in $U$, if there exists an analytic function $w(z)$ in $U$ such that $|w(z)| \leq |z|$, and $f(z) \equiv F(w(z))$, denoted by $f \prec F$ or $f(z) \sim F(z)$. If $F(z)$ is univalent in $U$, then the subordination is equivalent to $f(0) = F(0)$ and $f(U) \subset F(U)$.

Let $p, h \in H$ and let $\varphi(r,s,t;\cdot ) : C^3 \times U \to C$. If $p$ and $\varphi(p(z),zp'(z),z^2p''(z);z)$ are univalent and if $p$ satisfies the second-order superordination

$$h(z) \ p \ \varphi(p(z),zp'(z),z^2p''(z);z), \quad (3)$$

then $p$ is a solution of the differential superordination (3). (If $f$ is subordinate to $F$, then $F$ is superordinate to $f$.) An analytic function $q$ is called a subordinant if $q \prec p$ for all $p$ satisfying (3). A univalent subordinant $Q$ that satisfies $q \sim Q$ for all subordinants $q$ of (3) is said to be the best subordinant. Recently Miller and Mocanu [1] obtained conditions on $h, q$ and $\varphi$ for which the following implication holds:

$$h(z) \ p \ \varphi(p(z),zp'(z),z^2p''(z);z) \Rightarrow q(z) \ p \ p(z). \quad (4)$$

Using the results of Miller and Mocanu [1], Bulboacǎ [2] considered certain classes of first-order differential superordinations as well as superordination-preserving integral operators [3]. Ali et al. [4] have used the results of Bulboacǎ [2] and obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy
\[ q(z) p \frac{zf'(z)}{f(z)} p q_2(z), \]  
\[ q_1(z) p \frac{f(z)}{zf'(z)} p q_2(z) and q_1(z) p \frac{z^2f''(z)}{f^2(z)} p q_2(z) \]

where \( q_1 \) and \( q_2 \) are given univalent functions in \( U \) with \( q_1(0) = 1 \) and \( q_2(0) = 1 \). Shanmugam et al. [5] obtained sufficient conditions for normalized analytic functions \( f(z) \) to satisfy

\[ q_1(z) p \frac{f(z)}{zf'(z)} p q_2(z) \]

where \( q_1 \) and \( q_2 \) are given univalent functions in \( U \) with \( q_1(0) = 1 \) and \( q_2(0) = 1 \), while Obradović and Owa [6] obtained subordination results with the quantity \( (f(z)/z)^\alpha \) (see also [7]).

For \( 0 < \alpha < 1 \), a function \( f(z) \in N(\alpha) \) if and only if \( f(z) \in A \) and

\[ Re\left(\frac{zf'(z)}{f(z)} - \frac{z}{f(z)}\right) > 0, z \in U. \]  

\[ N(\alpha) \] was introduced by M.Obradović [8] recently, and he called this class of functions to be non-Bazilević type. Tuneski and Darus [9] obtained Fekete-Szegö inequality for the non-Bazilević class of functions. Using this non-Bazilević class, Wang et al. [10] studied many subordination results for the class \( N(\alpha, A, B) \) defined as

\[ N(\alpha, A, B) = \{ f(z) \in A : (1 + \lambda)\left(\frac{z}{f(z)}\right)^\alpha - \lambda \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)}\right)^\alpha p \frac{1 + Az}{1 + Bz}, z \in U \}. \]

where \( 0 < \alpha < 1, \lambda \in C, -1 \leq B \leq 1, A \neq B, A \in R \).

The main object of the present sequel to the aforementioned works is to apply a method based on the differential subordination in order to derive several subordination results. Furthermore, we obtain the previous results of Srivastava and Lashin [7], Singh [11], Shanmugam et al. [12] and Obradović and Owa [6] as special cases of some of the results presented here.

Some lemmas

To prove our main result, we will need the following definition and lemmas:

**DEFINITION**

[1] Denote by \( \Sigma \) the set of all functions \( f(z) \) that are analytic and injective on \( \overline{U} - E(f) \), where

\[ E(f) = \{ \xi \in \partial U : \lim_{z \to \xi} f(z) = \infty \}, \]

and are such that \( f'(\xi) \neq 0 \) for \( \xi \in \partial U - E(f) \).

**Lemma**

[5] Let \( q \) be univalent in \( U \) and let \( \beta, \gamma \in C \) with \( Re(1 + \frac{q'(\xi)}{q(\xi)}) > \max\{0, -Re \frac{\gamma}{\gamma} \} \). If \( p(z) \) is analytic in \( U \) and

\[ \beta p(z) + \gamma z p'(z) p \beta q(z) + \gamma z q'(z), \]

then \( p(z) p q(z) \) and \( q \) is the best dominant.

**Lemma**

[1] Let \( q \) be convex univalent in \( U \) and let \( \gamma \in C \) with \( Re(\gamma) > 0 \). If \( p(z) \in H[q(0), 1] \cap \Sigma \) and \( p(z) + \gamma z p'(z) \) is univalent in \( U \), and

\[ q(z) + \gamma z q'(z) p \ p(z) + \gamma z p'(z), \]

then \( q(z) p \ p(z) \) and \( q \) is the best subordinant.
Lemma
[13] Let \( q \) be univalent in \( U \) and let \( \theta, \rho \) be analytic in a domain \( \Omega \) containing \( q(U) \) with \( \rho(w) \neq 0 \) when \( w \in q(U) \). Set \( h(z) = zq'(z)\rho(q(z)) \). \( F(z) = \theta(q(z)) + h(z) \). Suppose that

1. \( h(z) \) is starlike univalent in \( U \);
2. \( \Re\left(\frac{F(z)}{h(z)}\right) > 0 \) for \( z \in U \).

If \( \theta(p(z)) + zq'(z)\rho(F(z))p \theta(q(z)) + zq'(z)\rho(q(z)), \) then \( p(z) \) and \( q(z) \) is the best dominant.

Lemma
[3] Let \( q \) be convex univalent in \( U \), and let \( \theta, \rho \) be analytic in a domain \( \Omega \) containing \( q(U) \). Suppose that

1. \( zq'(z)\rho(q(z)) \) is starlike univalent in \( U \);
2. \( \Re\left(\frac{h'(z)}{h(z)}\right) > 0 \) for \( z \in U \).

If \( p(z) \in \mathcal{H}[q(0), 1] \subseteq \Sigma \), with \( p(U) \subseteq \Omega \) and \( \theta(p(z)) + zp'(z)\rho(p(z)) \) is univalent in \( U \) and \( \theta(q(z)) + zq'(z)\rho(q(z)) \) is univalent in \( U \) and

\[ \theta(p(z)) + zp'(z)\rho(p(z)) \theta(q(z)) + zq'(z)\rho(q(z)), \]

then \( q(z) \) and \( q(z) \) is the best subordinant.

Sandwich theorems
By using Lemma 2.1, we first prove the following Theorem.

**Theorem 3.1.** Let \( q(z) \) be univalent in \( U \), \( 0 < \lambda < 1 \) and \( \alpha \in C \). Further assume that

\[ \Re\{1 + \frac{zq''(z)}{q'(z)}\} > \max\{0, -\Re\frac{\alpha}{\lambda}\}. \]

If \( f(z) \in A, g(z) \in S^* \), then

\[ \left(\frac{g(z)}{f(z)}\right) + \alpha z\left(\frac{g'(z)}{f(z)} - \frac{f'(z)}{f(z)}\right) \left(\frac{g(z)}{f(z)}\right) \]

implies that

\[ \left(\frac{g(z)}{f(z)}\right) \; q(z), \]

and \( q(z) \) is the best dominant.

**Proof.** Define the function \( p(z) \) by

\[ p(z) = \left(\frac{g(z)}{f(z)}\right), \]

Then a computation shows that

\[ \left(\frac{g(z)}{f(z)}\right) + \alpha z\left(\frac{g'(z)}{f(z)} - \frac{f'(z)}{f(z)}\right) \left(\frac{g(z)}{f(z)}\right) = p(z) + \frac{\alpha}{\lambda} zq'(z), \]

Then we obtain that

\[ p(z) + \frac{\nu}{\alpha} zp'(z) \; q(z) + \frac{\nu}{\alpha} zq'(z), \]

By using Lemma 2.1, we have the result.
Theorem 3.2. Let \( q(z) \) be convex univalent in \( U \), \( 0 < \lambda < 1 \), \( \alpha \in C \) with \( \text{Re}(\alpha) > 0 \). Suppose \((\frac{g(z)}{f(z)})^\alpha \in H[q(0), 1] \cap \Sigma \) and
\[
\left( \frac{g(z)}{f(z)} \right)^\lambda + \alpha z \left( \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)} \right) \left( \frac{g(z)}{f(z)} \right)^\lambda
\]
be univalent in \( U \). If
\[
q(z) + \frac{\gamma}{\alpha} z q'(z) p \left( \frac{g(z)}{f(z)} \right)^\lambda + \alpha z \left( \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)} \right) \left( \frac{g(z)}{f(z)} \right)^\lambda, \tag{20}
\]
then
\[
q(z) p \left( \frac{g(z)}{f(z)} \right)^\alpha \tag{22}
\]
and \( q(z) \) is the best subordinant.

Proof. Let \( p(z) = (\frac{g(z)}{f(z)})^\alpha \). Then Theorem 3.2 follows by an application of Lemma 2.2.

Combining the results of differential subordination and superordination, we obtain the following sandwich result.

Corollary 3.3. Let \( q_1(z) \) be univalent and let \( q_2(z) \) be convex univalent in \( U \), \( 0 < \lambda < 1 \) and \( \alpha \in C \) with \( \text{Re}(\alpha) > 0 \). Suppose \( q_2(z) \) satisfies (14). If \((\frac{g(z)}{f(z)})^\lambda \in H[q_1(0), 1] \cap \Sigma \), \( \left( \frac{g(z)}{f(z)} \right)^\lambda + \alpha z \left( \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)} \right) \left( \frac{g(z)}{f(z)} \right)^\lambda \) is univalent in \( U \), and
\[
q_1(z) + \frac{\alpha}{\lambda} z q_1(z) p \left( \frac{g(z)}{f(z)} \right)^\lambda + \alpha z \left( \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)} \right) \left( \frac{g(z)}{f(z)} \right)^\lambda p q_2(z) + \frac{\alpha}{\lambda} z q_2(z), \tag{23}
\]
then
\[
q_1(z) p \left( \frac{g(z)}{f(z)} \right)^\alpha \tag{24}
\]
and \( q_1(z) \) and \( q_2(z) \) are respectively the best subordinant and the best dominant.

For \( \alpha = 1 \) and \( g(z) = z \), we get the following corollary.

Corollary 3.4. Let \( q_1(z) \) be univalent and let \( q_2(z) \) be convex univalent in \( U \), \( 0 < \lambda < 1 \). Suppose \( q_2(z) \) satisfies (14). If \((\frac{g(z)}{f(z)})^\lambda \in H[q_1(0), 1] \cap \Sigma \), \( \left( \frac{g(z)}{f(z)} \right)^\lambda + \alpha z \left( \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)} \right) \left( \frac{g(z)}{f(z)} \right)^\lambda \) is univalent in \( U \), and
\[
q_1(z) + \frac{1}{\alpha} z q_1(z) p \left( 2 - \frac{z f'(z)}{f(z)} \right) \left( \frac{z}{f(z)} \right)^\lambda p q_2(z) + \frac{1}{\alpha} z q_2(z), \tag{25}
\]
then
\[
q_1(z) p \left( \frac{z}{f(z)} \right)^\alpha \tag{26}
\]
and \( q_1(z) \) and \( q_2(z) \) are respectively the best subordinant and best dominant.

4. Open Problem
Let \( H \) be the class of analytic functions in \( U = \{ z \in C : |z| < 1 \} \), and \( H[a, p] \) be the subclass of \( H \) consisting of functions of the form
\[
f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots. \tag{27}
\]
Let \( A(p) \) be the subclass of \( H \) consisting of functions of the form
\[
f(z) = z^p + a_{p+1} z^{p+1} + a_{p+2} z^{p+2} + \cdots. \tag{28}
\]
A function \( f \in A(p) \) is said to be in the class \( S^*(p) \) of \( p \)-valent starlike functions in \( U \), if it satisfies the inequality
\[
\Re\left(\frac{f''(z)}{f'(z)}\right) > 0, \quad z \in U.
\]
Let \( f(z) \in A(p) \) and \( g(z) \in S^*(p) \). We can consider sufficient conditions on \( h, q_1, q_2 \) and \( \varphi \) for which the following implication holds:
\[
q_1(z) p \left( \frac{g(z)}{f(z)} \right)^\alpha p q_2(z), \quad (29)
\]
or
\[
q_1(z) p \left( \frac{(1-\beta)f(z) + \beta zf'(z)}{g(z)} \right)^\alpha p q_2(z), \quad (30)
\]
where \( 0 < \alpha < 1 \) and \( 0 \leq \beta \leq 1 \).

REFERENCES