

Research Article

A new non-monotone line search technique

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Abstract: In this paper, we propose a new non-monotone line search algorithm for unconstrained optimization problems. We incorporate the proposed non-monotone strategy in [3] into an inexact Wolfe-type line search approach to construct a more relaxed line search procedure. The global convergence is subsequently proved under some mild classical assumptions.

Keywords: non-monotone, line search, Wolfe-type

INTRODUCTION

We consider the following unconstrained optimization problem

$$\min_{x \in R^n} f(x). \quad (1)$$

where $f(x)$ is twice continuously differentiable. Many iterative methods for (1) produce a sequence x_0, x_1, x_2, \dots where x_{k+1} is generated from x_k , the current direction d_k , and the step-size $\alpha_k > 0$ by the rule

$$x_{k+1} = x_k + \alpha_k d_k.$$

In monotone line search methods, α_k is chosen so that $f(x_{k+1}) < f(x_k)$. In 1982, Chamberlain et al. [2] proposed a watchdog technique for constrained optimization problems, in which some standard line search condition is relaxed to overcome the Maratos effect. Based on this idea, Grippo, Lampariello and Lucidi introduced a non-monotone line search technique for the Newton method in [1]. Their approach was roughly the following:

$$f(x_k + \alpha_k d_k) \leq f_{l(k)} + \delta \alpha_k g_k^T d_k. \quad (2)$$

where $\delta \in (0, 1)$ and

$$f_{l(k)} = \max_{0 \leq j \leq m(k)} \{f_{k-j}\}, k = 0, 1, 2, \dots \quad (3)$$

where $m(0) = 0$ and $0 \leq m(k) \leq \min\{m(k-1) + 1, N\}$ with $N \geq 0$. Non-monotone techniques [1, 3, 4, 5, 9, 10, 12] can improve convergence rate in the case that a monotone technique is forced to creep along the bottom of a narrow curved valley; also, they can improve the possibility of finding the global optimum. Encouraging numerical results have been reported [7, 8, 9, 11, 14].

Although the traditional non-monotone line search technique has many advantages, there are some drawbacks [3, 6, 12, 14]. In order to overcome those disadvantages, Ahoosh et al. [3] introduced a new formula instead of $f_{l(k)}$ in (2).

In detail, they define

$$R_k = \eta_k f_{l(k)} + (1 - \eta_k) f_k$$

where $0 \leq \eta_{\min} \leq \eta_{\max} \leq 1$ and $\eta_k \in [\eta_{\min}, \eta_{\max}]$.

In this article, we proposed a new line search algorithm for solving unconstrained optimization problems. In the algorithm, we combine a non-monotone strategy into a modified Wolfe-type rule and design a new algorithm that possibly chooses a larger step-length in each step. We define that

$$f(x_k + \alpha_k d_k) \leq R_k + \delta \alpha_k g_k^T d_k \tag{4}$$

$$f(x_k + \alpha_k d_k)^T d_k \geq \sigma g_k^T d_k \tag{5}$$

where d_k is a descent direction. This direction is determined by the following formula (see [15])

$$d_k = -B_k^{-1} g_k, \quad d_0 = g_0.$$

Where B_k is an approximation of the Hessian matrix at x_k and updated by Perry and Shanno formula:

$$B_{k+1} = \frac{\|y_k\|^2}{y_k^T s_k} I + \frac{y_k y_k^T}{y_k^T s_k} - \frac{\|y_k\|^2}{s_k^T y_k \|s_k\|^2} s_k s_k^T \tag{6}$$

where

$$y_k = g_{k+1} - g_k, \quad s_k = x_{k+1} - x_k.$$

Then the next iteration can be written as

$$d_{k+1} = -B_{k+1}^{-1} g_{k+1} = \frac{y_k^T s_k}{\|y_k\|^2} g_{k+1} + \left(\frac{y_k^T g_{k+1}}{\|y_k\|^2} - 2 \frac{s_k^T g_{k+1}}{\|y_k^T s_k\|} \right) s_k + \frac{s_k^T g_{k+1}}{\|y_k\|^2} y_k \tag{7}$$

The rest of this article is organized as follows: in Section 2, we describe a new non-monotone line search algorithm. In Section 3, we prove that the proposed algorithm is globally convergent. Finally, some conclusions are expressed in Section 4.

NEW ALGORITHM

Now, we can outline our new non-monotone line search algorithm is as follows:

Algorithm 1

Step 1 An initial point $x_0 \in R^n$ and symmetric matrix $B_0 \in R^{n \times n}$ are given. The constants $0 < \delta < \sigma < 1, 0 \leq \eta_{\min} \leq \eta_0 \leq \eta_{\max} \leq 1, N \geq 0$ and $\varepsilon > 0$ are also given. Set $k = 0, m(0) = 0$.

Step 2 Compute g_k . If $\|g_k\| < \varepsilon$, stop.

Step 3 Generate a descent direction d_k satisfying (7).

Step 4 Compute $f_{l(k)}$ and R_k , set $x_{k+1} = x_k + \alpha_k d_k$ where α_k satisfies (4) and (5).

Step 5 Update B_{k+1} according to (6). Set $k = k + 1$ and return to Step 2.

CONVERGENCE ANALYSIS

To prove the global convergence of the new algorithm, the following assumptions is proved throughout this paper:

(H1) The objective function f is continuously differentiable and has a lower bound on the level set $L(x_0) = \{x \in R^n \mid f(x) \leq f(x_0)\}$.

(H2) $g(x)$ of $f(x)$ is Lipschitz continuous function, there exists a constant $L > 0$ such that

$$\|g(x) - g(y)\| \leq L \|x - y\|, \quad \forall x, y \in R^n.$$

(H3) There exist positive constants c_1 and c_2 such that

$$g_k^T d_k \leq -c_1 \|g_k\|^2 \tag{8}$$

$$\|d_k\| \leq c_2 \|g_k\| \tag{9}$$

Lemma 1 (See Lemma 1 and Corollary 1 in [14]) Suppose that the sequence $\{x_k\}$ is generated by Algorithm 1 and (H1) and (8) holds. Then the sequence $\{f_{l(k)}\}$ is non-increasing and convergent.

Lemma 2 Suppose that the sequence $\{x_k\}$ is generated by Algorithm 1, (H1) and (H2) hold and the direction d_k satisfies (8) and (9). Then we have

$$\lim_{k \rightarrow \infty} f_{l(k)} = \lim_{k \rightarrow \infty} f(x_k) = \lim_{k \rightarrow \infty} R_k \tag{10}$$

Proof. The proof can be found Lemma 2 and Corollary 2 in [14].

Lemma 3 Assume that Algorithm 1 generates an infinite sequence $\{x_k\}$ and (H2) holds, then we have

$$\alpha_k \geq \left(\frac{1-\sigma}{L}\right) \frac{|g_k^T d_k|}{\|d_k\|^2}. \tag{11}$$

Proof. From (5) and (H2), we have

$$L\alpha_k \|d_k\|^2 \geq (g(x_k + \alpha_k d_k) - g(x_k))^T d_k \geq (\sigma - 1)g_k^T d_k > 0. \tag{12}$$

Thus, we can conclude that

$$\alpha_k \geq \frac{(\sigma - 1)g_k^T d_k}{L\|d_k\|^2} = \left(\frac{1-\sigma}{L}\right) \frac{|g_k^T d_k|}{\|d_k\|^2}. \tag{13}$$

This completes the proof.

Theorem 4 Suppose that the sequence $\{x_k\}$ is generated by Algorithm 1 and (H1), (H2) and (H3) hold. Then we have

$$\lim_{k \rightarrow \infty} \|g_k\| = 0. \tag{14}$$

Proof. We first show that

$$f_{k+1} \leq R_k - \beta \|g_k\|^2 \tag{15}$$

Where

$$\beta = \frac{\delta(1-\sigma)c_1^2}{Lc_2^2} \tag{16}$$

Using (4) and Lemma 3, we get

$$f_{k+1} \leq R_k + \delta\alpha_k g_k^T d_k \leq R_k - \left(\frac{\delta(1-\sigma)}{L}\right) \left(\frac{g_k^T d_k}{\|d_k\|}\right)^2. \tag{17}$$

From (8) and (9), we obtain

$$f_{k+1} \leq R_k - \left(\frac{\delta(1-\sigma)c_1^2}{Lc_2^2}\right) \|g_k\|^2. \tag{18}$$

This indicates that (15) holds.

By setting β as (16), it follows that $\beta > 0$. Also by (15), we have

$$R_k - f_{k+1} \geq \beta \|g_k\|^2 \geq 0.$$

This fact along with (10) give (14). This completes the proof.

Theorem 5 Suppose that the sequence $\{x_k\}$ is generated by Algorithm 1 and (H1), (H2) and (H3) hold, then there is no limit point of the sequence $\{x_k\}$ being a local maximum of $f(x)$.

Proof. The proof of this fact is similar to the proof given by Grippo et al. in [1], hence we omit the details.

CONCLUSIONS

In this paper, we propose a new non-monotone Wolfe-type line search algorithm for solving unconstrained optimization problems. After we analyzed the properties of the new algorithm, the global convergence theory is proved. We believe that there is considerable scope for modifying and adapting the basic ideas introduced in this paper. In the near future, we would like to combine the new algorithm with trust region algorithm in order to sufficiently use the information which the algorithm has already derived.

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