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Research Article

Application of Uniform Continuity Test for Function

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Abstract: The uniform continuity of the function is an important basic theory of the mathematical analysis course, and it has important affect on the future courses. Therefore, it is of great importance to judge the uniform continuity. This paper provides eight practical judgment methods and Three comparative judgment methods. Besides, through some examples, it emphasizes judging the uniform continuity of the function, and selects proper judgment methods according to the characteristics of the function and interval.
Keywords: continuity, uniform continuity, differentiable function

BASIC JUDGMENT METHODS FOR THE UNIFORM CONTINUITY OF FUNCTION

Theorem 1[1-3] Set function f to be continuous in the finite interval $(a,b]([a,b))$, so the necessary and sufficient condition for f to be uniformly continuous in $(a,b]([a,b))$ is: $\lim_{x\to a^+} f(x)(\lim_{x\to b^-} f(x))$ exists.

Example 1 $f(x)=(x-a)^{\lambda}, x\in(a,b]$. $\lim_{x\to a^+} f(x)=\begin{cases} 0, \lambda>0, \\ 1, \lambda=0, \\ +\infty, \lambda<0. \end{cases}$

From theorem 1, when $\lambda\geq 0$, f is uniformly continuous in $(a,b]$; when $\lambda<0$, f is not uniformly continuous in $(a,b]$.

Example 2 $f(x)=\log_c(x-a)$, $x\in(a,b]$, among which, $c>0$, $c\neq 1$. $\lim_{x\to a^+} f(x)=-\infty$. From theorem 1, f is not uniformly continuous in $(a,b]$.

Theorem 2[1-3] Set function f to be continuous in the infinite interval $[a,+\infty)((-\infty,b])$. If $\lim_{x\to+\infty} f(x)(\lim_{x\to-\infty} f(x))$ exists, f will be uniformly continuous in $[a,+\infty)((-\infty,b])$.

Example 3 $f(x)=\dfrac{\sin x}{x}$, $x\in(0,+\infty)$.

$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\dfrac{\sin x}{x}=1$. From theorem 1, $f(x)=\dfrac{\sin x}{x}$ is uniformly continuous in $(0,1]$.

$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\dfrac{\sin x}{x}=0$. From theorem 2, $f(x)=\dfrac{\sin x}{x}$ is uniformly continuous in $[1,+\infty)$.

So, $f(x)=\dfrac{\sin x}{x}$ is uniformly continuous in $(0,+\infty)$.

Theorem 3 If function f is the continuous periodic function in $(-\infty,+\infty)$, f is uniformly continuous in $(-\infty,+\infty)$.

From theorem 3, $\sin kx$和$\cos kx$ is uniformly continuous in $(-\infty,+\infty)$.

Theorem 4 If the monotonous bounded function f is continuous in infinite or finite interval (a,b), f is uniformly continuous in (a,b).

From **t**heorem 4, $\arcsin x$ and $\arccos x$ is uniformly continuous in $[-1,1]$, $\arctan x$ and $arc\cot x$ is uniformly continuous in $(-\infty,+\infty)$.

Theorem 5[4] The necessary and sufficient condition for Function f to be uniformly continuous in the interval I is: $\forall\{x_n\},\{x_n'\}\subset I,\ x_n - x_n' \to 0\ (n\to\infty)$, and then,

$$\lim_{n\to\infty}[f(x_n)-f(x_n')]=0.$$

Example 4 $f(x)=e^x, x\in(0,+\infty)$.

Select $x_n=\ln(n+1), x_n'=\ln n, n=1,2,\Lambda$. Meet $\lim\limits_{n\to\infty}(x_n-x_n')=\lim\limits_{n\to\infty}\ln(1+\frac{1}{n})=0.$

But, $\lim\limits_{n\to\infty}[f(x_n)-f(x_n')]=\lim\limits_{n\to\infty}[n+1-n]=1\neq 0.$ So, f is not uniformly continuous in $(0,+\infty)$.

Theorem 6 If function f and g are uniformly continuous in the interval I, $f\pm g$ is also uniformly continuous in the interval I; when I is a finite interval, $f\cdot g$ is also uniformly continuous in I.

Example 5 $f(x)$ is uniformly continuous in $[a,+\infty)$, and $\lim\limits_{x\to+\infty}[f(x)-bx-c]=l.$

From theorem 2, $f(x)-bx-c$ is uniformly continuous in $[a,+\infty)$. Besides, $bx+c$ is uniformly continuous in $[a,+\infty)$, From Theorem 6, $f(x)$ is uniformly continuous in $[a,+\infty)$.

Theorem 7 If function f can be derived in the interval I, and f' has boundary, f is uniformly continuous in I.

From theorem 7, $\sin kx$, $\cos kx$, $\arctan x$ and $arc\cot x$ are uniformly continuous in $(-\infty,+\infty)$.

Theorem 8 Set f to be continuous in $[a,+\infty)$, and can be derived in $(a,+\infty)$, and $\lim\limits_{x\to+\infty}\left|f'(x)\right|=A$ (normal extremity or abnormal extremity $+\infty$ exists), the necessary and sufficient condition for f to be uniformly continuous in $[a,+\infty)$ is: $A<+\infty$.

Example 6 $f(x)=\log_a x, x\in[c,+\infty)$, among which, $a>0$, and $a\neq 1$, $c>0.$

$f'(x)=\dfrac{1}{x\ln a}$. $\lim\limits_{x\to+\infty}\left|f'(x)\right|=0$. From theorem 8, $f(x)=\log_a x$ is uniformly continuous in $[c,+\infty)$.

Example 7 $f(x)=x^\lambda, x\in[a,+\infty)$, among which, $a>0$. $f'(x)=\lambda x^{\lambda-1}$, $\lim\limits_{x\to+\infty}f'(x)=\begin{cases}0, \lambda<1,\\ 1, \lambda=1,\\ +\infty, \lambda>1.\end{cases}$

From theorem 8, when $\lambda\leq 1$, $f(x)=x^\lambda$ is uniformly continuous in $[a,+\infty)$; when $\lambda>1$, $f(x)=x^\lambda$ is not uniformly continuous in $[a,+\infty)$.

COMPARATIVE JUDGMENT METHOD FOR THE UNIFORM CONTINUITY OF THE FUNCTION

Theorem 9 Set function f and g to be derived in I, if $L>0,$ it achieves $\forall x\in I,$ $\left|g^{/}(x)\right|\le L\left|f^{/}(x)\right|$, So,

(1) When f is uniformly continuous in the interval I, g is also uniformly continuous in the interval I.

(2) When g is not uniformly continuous in the interval I, f will be not uniformly continuous in the interval I.

Example 8 $f(x)=\sin\sqrt{x}, x\in[1,+\infty).$

Record $g(x)=\sqrt{x}.$ From example 7, $g(x)=\sqrt{x}$ is uniformly continuous in $[1,+\infty)$.

$\left|f^{/}(x)\right|=\left|\dfrac{\cos\sqrt{x}}{2\sqrt{x}}\right|\le\left|g^{/}(x)\right|.$ From theorem 9, $f(x)=\sin\sqrt{x}$ is uniformly continuous in $[1,+\infty)$.

Example 9 $f(x)=\sqrt{x}e^{x}, x\in[1,+\infty).$

Record $g(x)=e^{x}.$ From example 4, $g(x)=e^{x}$ is not uniformly continuous in $[1,+\infty)$.

$\left|f^{/}(x)\right|=e^{x}(\dfrac{1}{2\sqrt{x}}+\sqrt{x})\ge e^{x}=\left|g^{/}(x)\right|.$ From theorem 9, $f(x)=\sqrt{x}e^{x}$ is not uniformly continuous in $[1,+\infty)$.

Theorem 10[5] Set function f and g to be continuous in $[a,+\infty)$, and can be derived in $(a,+\infty)$, and $\lim\limits_{x\to+\infty}\left|\dfrac{f^{/}(x)}{g^{/}(x)}\right|=l,$ So,

(1) When $0<l<+\infty$, as for f and g, if anyone is uniformly continuous in $[a,+\infty)$, and the other one is uniformly continuous in $[a,+\infty)$. One of them is uniformly continuous in $[a,+\infty)$, and the other one is uniformly continuous in $[a,+\infty)$.

(2) When $l=0$, if g is uniformly continuous in $[a,+\infty)$, 则 f is also uniformly continuous in $[a,+\infty)$;

(3) when $l=+\infty$, if g is not uniformly continuous in $[a,+\infty)$, f will not be uniformly continuous in $[a,+\infty)$.

Example10 $f(x)=\ln(x+\sqrt{x}), x\in[1,+\infty).$

Record $g(x)=\ln x.$ From example 6, $g(x)=\ln x$ is uniformly continuous in $[1,+\infty)$.

$\lim\limits_{x\to+\infty}\left|\dfrac{f^{/}(x)}{g^{/}(x)}\right|=\lim\limits_{x\to+\infty}\dfrac{x+\frac{1}{2}\sqrt{x}}{x+\sqrt{x}}=1.$ From theorem 10, $f(x)=\ln(x+\sqrt{x})$ is uniformly continuous in $[1,+\infty)$.

Example 11 $f(x)=x^{2}+x+1, x\in[1,+\infty).$

记$g(x)=x^{2}.$ From example **7**, $g(x)$ is not uniformly continuous in $[1,+\infty)$.

$\lim\limits_{x\to+\infty}\left|\dfrac{f^{/}(x)}{g^{/}(x)}\right|=\lim\limits_{x\to+\infty}\dfrac{2x+1}{2x}=1.$ From theorem 10, $f(x)=x^{2}+x+1$ is not uniformly continuous in $[1,+\infty)$.

Theorem 11[5] Set function f to be continuous in $[a,+\infty)$, and can be derived from $(a,+\infty)$, $a>0,$ and $\lim\limits_{x\to+\infty}x^{\lambda}\left|f^{/}(x)\right|=l,$ So,

(1) When $\lambda\ge 0$, and $0\le l<+\infty$, f is uniformly continuous in $[a,+\infty)$;

(2) When $\lambda<0$, and $0<l\le+\infty$时，f is not uniformly continuous in $[a,+\infty)$.

Example 12 $f(x)=\sqrt[3]{x^{2}+x+1}, x\in[1,+\infty).$

$$\lim_{x\to+\infty} x^{\frac{1}{3}}\left|f'(x)\right| = \lim_{x\to+\infty}\frac{2x^{\frac{4}{3}}+x^{\frac{1}{3}}}{3\sqrt[3]{(x^2+x+1)^2}} = \frac{2}{3}.$$

$\lambda = \frac{1}{3} > 0, l = \frac{2}{3}$. **F**rom theorem 11, $f(x)$ is uniformly continuous in $[1,+\infty)$.

Example 13 $f(x) = x^3 + \sqrt{x}, x \in [1,+\infty).$

$\lim_{x\to+\infty} x^{-1}\left|f'(x)\right| = \lim_{x\to+\infty}\frac{3x^2+\frac{1}{2\sqrt{x}}}{x} = +\infty.$ $\lambda = -1 > 0, l = +\infty.$ **F**rom theorem 11, $f(x)$ is not uniformly continuous in $[1,+\infty)$.

CONCLUSIONS

This paper provides 8 practical basic judgment methods and 3 comparative judgment methods. Besides, it also provides some examples, and shows it is quite effective to select proper judgment methods according to the characteristics of the function and interval.

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