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Research Article

Cubic Diophantine Equation With Three Unknowns $(a+3)x^2-ay^2=27z^3$

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Abstract: The non homogeneous ternary cubic diophantine equation given by $(a+3)x^2-ay^2=27z^3$ is considered. Different patterns of non-zero distinct integer solutions to the above equation are obtined when a=1, 2 and 5 . For each of there cases, a few interesting properties between the solutions and special numbers are presented.
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INTRODUCTION:

Integral solutions for the homogeneous or non-homogeneous Diophantine cubic equation is an interesting concept as it can seen from [1-3]. In [4-10] a few special cases of cubic Diophantine equation with four unknowns are studied. In [11-13] cubic equations with five unknowns are studied for their integral solutions. In [14, 15] cubic equation with six unknowns are studied for their integral solutions. This communication concerns with yet another cubic Diophantine equation with three unknowns $(a+3)x^2-ay^2=27z^3$. A few relations among the solutions are presented.

Notations:

$T_{m,n}$: Polygonal number of rank n with m sides

P_n^m : Pyramidal number of rank n with m sides

Pr_n : Pronic number of rank n

S_n : Star number of rank n

OH_n : Octahedral number of rank n

SO_n : Stella Octangular number of rank n

$CP_{m,n}$: Centered Pyramidal number of rank n with sides m

METHOD OF ANALYSIS:

The Ternary cubic Diophantine equation to be solved is given by

$$(a+3)x^2-ay^2=27z^3 \quad (1)$$

The substitution of the linear transformation

$$x=X+aT,\ y=X+(a+3)T \quad (2)$$

in (1) leads to $X^2-a(a+3)T^2=9z^3$ (3)

Assume $z(\alpha,\beta,a)=\alpha^2-a(a+3)\beta^2,\ \alpha,\beta>0$ (4)

Write 9 as $9=[(2a+3)+2\sqrt{a(a+3)}][(2a+3)-2\sqrt{a(a+3)}]$ (5)

Substituting (4) and (5) in (3) and employing the method of factorization, define

$$(X+\sqrt{a(a+3)}T)=((2a+3)+2\sqrt{a(a+3)})(\alpha+\sqrt{a(a+3)}\beta)^3$$

Equating the rational and irrational parts on both sides, we get

$$X = (2a+3)[\alpha^3 + 3a(a+3)\beta^2\alpha] + 2a(a+3)[3\alpha^2\beta + a(a+3)\beta^3]$$

$$T = (2a+3)[3\alpha^2\beta + a(a+3)\beta^3] + 2[\alpha^3 + 3a(a+3)\beta^2\alpha]$$

Substituting the values of X and T in (2), we get

$$x(\alpha,\beta,a) = (4a+3)[\alpha^3 + 3a(a+3)\beta^2\alpha] + a(4a+9)[3\alpha^2\beta + a(a+3)\beta^3]$$
$$y(\alpha,\beta,a) = (4a+9)[\alpha^3 + 3a(a+3)\beta^2\alpha] + (4a^2+15a+9)[3\alpha^2\beta + a(a+3)\beta^3] \quad (6)$$

Thus (4) and (6) represent non-zero distinct integral solutions of (1).

Properties:

- $x(\alpha,\beta,a) - y(\alpha,\beta,a) \equiv 0 \pmod{3T}$
- $\{(a+3)x^2(\alpha,\beta,a) - ay^2(\alpha,\beta,a)\}$ is a cubical integer.
- $6z(1,2,-1)$ is a Nasty number.

To analyse the nature of solution one has to go in for particular values of a, in equation (1). For the sake of simplicity and clear understanding we exhibit below the integer solutions of (1) along with the properties for the cases a=1,a=3 and a=5.

Case: 1

Let a=1,

(3) becomes $X^2 - 4T^2 = 9z^3$ (7)

which is equivalent to $\begin{matrix}(X+2T)=9z^2\\(X-2T)=z\end{matrix}$, $\begin{matrix}(X+2T)=3z^2\\(X-2T)=3z\end{matrix}$, $\begin{matrix}(X+2T)=z^2\\(X-2T)=9z\end{matrix}$

solving these equations, we get the value of X and T. In view of (2), the corresponding integral solutions of (1) are

Choice :1	Choice :2	Choice :3
$x = 108A^2 + A$	$x = 24A^2 + 6A$	$x = 12A^2 + 9A$
$y = 216A^2 - 2A$	$y = 12A^2 - 3A$	$y = 24A^2 - 18A$
$z = 4A$	$z = 4A$	$z = 4A$

Case: 2

Take a=2,

(3) becomes $X^2 - 10T^2 = 9z^3$ (8)

Assume $z(\alpha,\beta) = \alpha^2 - 10\beta^2,\ \alpha,\beta > 0$ (9)

Illustration-1:

Write 9 as $9 = (7+2\sqrt{10})(7-2\sqrt{10})$ (10)

Substituting (9) and (10) in (8) and employing the method of factorization, define

$$(X + \sqrt{10}T) = (7+2\sqrt{10})(\alpha + \sqrt{10}\beta)^3$$

Equating the rational and irrational parts on both sides, we get

$$X = 7\alpha^3 + 210\beta^2\alpha + 60\alpha^2\beta + 200\beta^3$$

$$T = 2\alpha^3 + 60\beta^2\alpha + 21\alpha^2\beta + 70\beta^3$$

Substituting the values of X and T in (2), we get

$$x(\alpha,\beta) = 11\alpha^3 + 330\beta^2\alpha + 102\alpha^2\beta + 340\beta^3$$
$$y(\alpha,\beta) = 17\alpha^3 + 510\beta^2\alpha + 165\alpha^2\beta + 550\beta^3 \quad (11)$$

Thus (9) and (11) represent non-zero distinct integral solutions of (1).

Properties:

- $11y(\alpha,\alpha+1)-17x(\alpha,\alpha+1)=162P_{\alpha}^{5}-270CP_{\alpha}^{6}$
- $\{55x(\beta+1,\beta)-34y(\beta+1,\beta)-1620P_{\beta}^{5}\}$ is a cubical integer.
- $55x(\alpha,\alpha)-34y(\alpha,\alpha)-z(\alpha,\alpha)\equiv 0(\bmod 9\alpha^{2})$

Illustration-2:

Write 9 as $9=(13+4\sqrt{10})(13-4\sqrt{10})$

Repeating the above process the non-zero distinct integral solutions of (1) are

$$x(\alpha,\beta)=21\alpha^{3}+630\beta^{2}\alpha+198\alpha^{2}\beta+660\beta^{3}$$

$$y(\alpha,\beta)=33\alpha^{3}+990\beta^{2}\alpha+315\alpha^{2}\beta+1050\beta^{3}$$

$$z(\alpha,\beta)=\alpha^{2}-10\beta^{2}$$

Properties:

- $21y(2\alpha+1,1)-33x(2\alpha+1,1)=Ky_{\alpha}+432$
- $105x(\alpha,\beta)-66y(\alpha,\beta)+z(\alpha,\alpha)+t_{4,3\alpha}\equiv 0(\bmod 27)$
- $21y(\alpha,\alpha)-33x(\alpha,\alpha)-162CP_{\alpha}^{3}\equiv 0(\bmod 27)$

Illustration-3:

Write 9 as $9=(57+18\sqrt{10})(57-18\sqrt{10})$

Repeating the above process as in illustration-1, the corresponding non-zero distinct integral solutions of (1) are

$$x(\alpha,\beta)=93\alpha^{3}+2790\beta^{2}\alpha+882\alpha^{2}\beta+2940\beta^{3}$$

$$y(\alpha,\beta)=147\alpha^{3}+4410\beta^{2}\alpha+1395\alpha^{2}\beta+4650\beta^{3}$$

$$z(\alpha,\beta)=\alpha^{2}-10\beta^{2}$$

Properties:

- $93y(t_{3,\alpha},t_{3,\alpha+2})-147x(t_{3,\alpha},t_{3,\alpha+2})=3888(t_{3,\alpha}*Pt_{\alpha})+270(Pr_{\alpha+2})^{3}$
- $\{5x^{2}(\alpha,\beta)-2y^{2}(\alpha,\beta)\}$ is a cubical integer.
- $9y(\alpha,\alpha)-147x(\alpha,\alpha)=13CP_{3\alpha}^{6}$

Case: 3

Let a=5,

(3) becomes $X^{2}-40T^{2}=9z^{3}$ (12)

Assume $z(\alpha,\beta)=\alpha^{2}-40\beta^{2}$, $\alpha,\beta>0$ (13)

Illustration-1:

Write 9 as $9=(7+\sqrt{40})(7-\sqrt{40})$ (14)

Substituting (13) and (14) in (12) and employing the method of factorization, define

$$(X+\sqrt{0}T)=(7+\sqrt{40})(\alpha+\sqrt{40}\beta)^{3}$$

Equating the rational and irrational parts on both sides, we get

$$X=7\alpha^{3}+840\beta^{2}\alpha+120\alpha^{2}\beta+1600\beta^{3}$$

$$T=\alpha^{3}+120\beta^{2}\alpha+21\alpha^{2}\beta+280\beta^{3}$$

Substituting the values of X and T in (2), we get

$$x(\alpha,\beta)=12\alpha^3+1440\beta^2\alpha+225\alpha^2\beta+3000\beta^3$$
$$y(\alpha,\beta)=15\alpha^3+1800\beta^2\alpha+288\alpha^2\beta+3840\beta^3 \quad (15)$$

Thus (15) and (13) represent non-zero distinct integral solutions of (1).

Properties:

- $12y(\alpha,\alpha+1)-15x(\alpha,\alpha+1)-162P_\alpha^5\equiv 0(\mathrm{mod}\,1080)$
- $\{24y(1,\alpha(\alpha-1))-30x(1,\alpha(\alpha-1))=27(S_\alpha-1)+2160(Pr_{\alpha-1})^3\}$
- $96x(\alpha(2\alpha^2-1),1)-75y(\alpha(2\alpha^2-1),1)-3240SO_\alpha$ is a cubical integer.

Illustration-2:

Write 9 as $9=(13+2\sqrt{40})(13-2\sqrt{40})$

Repeating the above process the non-zero distinct integral solutions of (1) are

$$x(\alpha,\beta)=23\alpha^3+2760\beta^2\alpha+435\alpha^2\beta+5800\beta^3$$
$$y(\alpha,\beta)=29\alpha^3+3480\beta^2\alpha+552\alpha^2\beta+7360\beta^3$$
$$z(\alpha,\beta)=\alpha^2-40\beta^2$$

Properties:

- $23y(1,\alpha(2\alpha^2+1))-29x(1,\alpha(2\alpha^2+1))-243OH_\alpha\equiv 0(\mathrm{mod}\,540)$
- $\{184x(1,\alpha(\alpha+1))-145x(1,\alpha(\alpha+1))-6480t_{3,\alpha}\}$ is a cubical integer.
- $8x^2(\alpha,\beta)-5y^2(\alpha,\beta)\equiv 0(\mathrm{mod}\,27)$

CONCLUSION:

In this paper, we have presented four different patterns of non-zero distinct integer solutions of the cubic Diophantine equation given by $(a+3)x^2-ay^2=27z^3$. To conclude, one may search for other patterns of solutions and their corresponding properties.

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