Existence of Positive Solutions of Nonlinear Fourth-order Boundary Problem with Parameter

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Abstract: This paper is concerned with the fourth-order boundary problem

\[
\begin{align*}
  u^{(4)}(t) - \rho^4 u(t) &= f(t, u(t)), \\
  u(0) &= 0, u(1) = 0, \\
  u^{(n)}(0) &= 0, u^{(n)}(1) = \lambda
\end{align*}
\]

where \(\lambda > 0\) and \(0 < \rho < \frac{\pi}{2}\) is a parameter. \(f : [0, 1] \times [0, +\infty) \to \mathbb{R}\) is a nonnegative and continuous function. Combine with the properties of the Green function using Fixed Point theorem in cones, proved the existence of positive solutions nonlinear fourth-order boundary value problem

Keywords: Fourth-order Boundary value problem, one, Positive solutions, Fixed point

MSC: 34B10, 34B15

INTRODUCTION

In this paper, we think of the nonlinear fourth-order boundary value problems (BVP for short)

\[
\begin{align*}
  u^{(4)}(t) - \rho^4 u(t) &= f(t, u(t)), \\
  u(0) &= 0, u(1) = 0, u^{(n)}(0) &= 0, u^{(n)}(1) = \lambda
\end{align*}
\]

where \(\lambda > 0\) and \(0 < \rho < \frac{\pi}{2}\), \(f : [0, 1] \times [0, +\infty) \to \mathbb{R}\) is a nonnegative and continuous function.

Function \(u(t)\) which is positive on \((0, 1)\) and \(u(t) \in C^1[0,1] \cap C^4[0,1]\), if \(u(t)\) satisfied differential equation (1.1) and the boundary conditions (1.2), we call it the positive solution of the nonlinear fourth-order boundary problem of (1.1). It is assumed throughout that

\(H_1:\) \(f(t,u)\) is integral for each fixed \(u \in [0,1] \times [0,+\infty)\), and \(0 < \int_0^1 f(t, u(t)) dt < +\infty\);

\(H_2:\) \(\limsup\limits_{u \to +\infty, t \in [0,1]} \frac{f(t,u)}{u} = 0, \liminf\limits_{u \to +\infty, t \in [0,1]} \frac{f(t,u)}{u} = \infty\);

\(H_3:\) \(\limsup\limits_{u \to +\infty, t \in [0,1]} \frac{f(t,u)}{u} = \infty, \liminf\limits_{u \to +\infty, t \in [0,1]} \frac{f(t,u)}{u} = 0\).

It is well-known that the fixed point theorem method is a powerful tool for proving the existence results for boundary value problem (BVP for short). It has been used to deal with the multi-point BVP for second-order ordinary differential equations and the two-point BVP for higher-order ordinary differential equations, see [1-4]. But there are fewer results on multi-point higher-order BVPs in the literature. In 2006, by using the upper and lower solution method, the authors studied the following fourth-order four-point BVP[5].
\[ u^{(4)}(t) = f(t,u(t),u'(t)), t \in [0,1] = I \]
\[ u(0) = 0, u(1) = 0 \]
\[ a u''(\xi_1) - b u''(\xi_1) + c u''(\xi_2) + du''(\xi_2) = 0, \]

They obtained the existence results for BVP under the condition \( f(t,u,v) \) is increasing on \( u \) and decreasing on \( v \), i.e.

\[ f(t,u_2,v) - f(t,u_1,v) \geq 0, u_1 \leq u_2 \]
\[ f(t,u,v_2) - f(t,u,v_1) \leq 0, v_1 \leq v_2 \]  

De-Xiang Ma and Xiao-Zhong Yang [4] by using the upper and lower solution method, proved the fourth-order four-point boundary value problem

\[ f(t,u,v) \] is weak-increasing on \( u \) and weak-decreasing on \( v \). They gave a critical theorem, a new maximum principle. Inspired and motivated by the works mentioned, we study a group of contains parameter of nonlinear fourth-order boundary value problems, proved the existence of positive solution.

**Preliminary**

In this section, we will give some preliminary considerations and some lemmas which are essential to our main result.

**Lemma 2.1:** Assume \( m,n,q \) are constants, \( \varphi_1(t), \varphi_2(t) \) are two independent solutions of the non-homogeneous equation \( m \varphi''(t) + n \varphi'(t) + q \varphi(t) = h(t) \), \( \varphi_0(t) \) is one of the solutions of the boundary problem \((2.1)\) from the general solution of non-homogeneous equation, we can get \( \varphi(t) = c_1 \varphi_1(t) + c_2 \varphi_2(t) + \varphi_0(t) \) is the general solution of the equation \( a \varphi''(t) + b \varphi'(t) + c \varphi(t) = h(t) \), where \( c_1, c_2 \) are any two constants.

\[
\begin{cases}
  m \varphi''(t) + n \varphi'(t) + q \varphi(t) = h(t) \\
  \varphi(0) = 0, \varphi(1) = 0.
\end{cases}
\]

**Proof:** It can be validation directly by the structure of non-singular equation.

Consider the nonlinear second order boundary problem first.

\[
\begin{cases}
  u''(t) - \rho^2 u(t) = -v(t) \\
  u(0) = 0, u(1) = 0
\end{cases}
\]

It is easily to compute \((2.2)\) is equivalent to the following integral equation

\[ u(t) = \int_0^1 G(t,s)v(s)ds, \]

Where

\[ G(t,s) = \begin{cases}
    \frac{\sinh(\rho s)\sinh(\rho - pt)}{\rho \sinh(\rho)}, & 0 \leq s \leq t \leq 1 \\
    \frac{\sinh(\rho t)\sinh(\rho - ps)}{\rho \sinh(\rho)}, & 0 \leq t \leq s \leq 1
  \end{cases} \]

Consider the nonlinear second order boundary problem
\[
\begin{aligned}
v''(t) + \rho^2 v(t) &= -f(t, u(t)) \\
v(0) &= 0, v(1) = \lambda
\end{aligned}
\] (2.5)

we have already known the nonlinear second order boundary problem
\[
\begin{aligned}
v''(t) + \rho^2 v(t) &= -f(t, u(t)) \\
v(0) &= 0, v(1) = 0
\end{aligned}
\]
is equivalent to the following integral equation
\[
v(t) = \int_0^1 G_2(t, s) f(s, u(s)) ds,
\]
where
\[
G_2(t, s) = \begin{cases}
\frac{\sin \rho s \sin \rho (1-t)}{\rho \sin \rho}, & 0 \leq s \leq t \leq 1 \\
\frac{\sin \rho t \sin \rho (1-s)}{\rho \sin \rho}, & 0 \leq t \leq s \leq 1
\end{cases}
\] (2.6)

And because of \(\varphi_1(t) = \cos(\rho t), \varphi_2(t) = \sin(\rho t)\) are two independent solutions of equation
\[
v''(t) + \rho^2 v(t) = 0,
\]
from lemma 2.1, we can say the general solution of boundary problem (2.5) can be represented
\[
v(t) = c_1 \cos(\rho t) + c_2 \sin(\rho t) + \int_0^t G_2(t, s) f(s) ds, \]
also satisfy the conditions \(v(0) = 0, v(1) = \lambda\), according to this boundary condition we can calculate the coefficient of \(c_1, c_2\), after computing and tidying, the existence of boundary problem (2.5) can use the following integral equation
\[
v(t) = \frac{\lambda \sin \rho t}{\sin \rho} + \int_0^t G_2(t, s) f(s, u(s)) ds
\] (2.7)

put (2.7) into (2.3), we receive the solution of the nonlinear boundary problem
\[
u(t) = \frac{\lambda \sin \rho s}{\sin \rho} \int_0^t G_1(t, s) ds + \int_0^t \int_0^t G_1(t, s) G_2(s, \tau) f(\tau, u(\tau)) d\tau ds
\]

Lemma 2.2: For all \((s, t) \in [0, 1] \times [0, 1]\), we have
\[
G_1(t, s) = \begin{cases}
\frac{\sinh(\rho - \rho t)}{\sinh(\rho - \rho s)}, & 0 \leq s \leq t \leq 1 \\
\frac{\sinh(\rho t)}{\sinh(\rho s)}, & 0 \leq t \leq s \leq 1
\end{cases}
\]
\[
\rho t (1-t) \text{csch}（\rho）G_1(s, s) \leq G_1(t, s) \leq G_1(s, s)
\]

Proof: It is clearly to see
\[
G_1(t, s) = \begin{cases}
\frac{\sinh(\rho - \rho t)}{\sinh(\rho - \rho s)}, & 0 \leq s \leq t \leq 1 \\
\frac{\sinh(\rho t)}{\sinh(\rho s)}, & 0 \leq t \leq s \leq 1
\end{cases}
\]
It is obvious that \( G_i(t,s) \leq G_i(s,s) \). The proof is complete.

Define an integral operator \( \Phi : C^+ [0,1] \to C^+ [0,1] \) by

\[
\Phi u(t) = \frac{\lambda(t)}{\sin \theta} \int_0^1 G_i(t,s) \, ds + \int_0^1 \int_0^1 G_i(t,s)G_2(s,\tau) f(\tau, u(\tau)) \, d\tau \, ds
\]

(2.8)

Then, only if nonzero fixed point \( u(t) \) of mapping \( \Phi \) defined by (2.8) is a positive solution of (1.1) and (1.2)

**Lemma 2.3:** \( \Phi(K) \subset K \)

**Proof:** For any \( u \in K \), from lemma 2.2 we have

\[
\|\Phi u(t)\| = \max \frac{\lambda(t)}{\sin \theta} \int_0^1 G_i(t,s) \, ds + \int_0^1 \int_0^1 G_i(t,s)G_2(s,\tau) f(\tau, u(\tau)) \, d\tau \, ds
\]

And inequalities

\[
\|\Phi u(t)\| \leq \max \frac{\lambda(t)}{\sin \theta} \int_0^1 G_i(t,s) \, ds + \int_0^1 \int_0^1 G_i(t,s)G_2(s,\tau) f(\tau, u(\tau)) \, d\tau \, ds
\]

\[
\min \frac{\lambda(t)}{\sin \theta} \int_0^1 G_i(t,s) \, ds + \int_0^1 \int_0^1 G_i(t,s)G_2(s,\tau) f(\tau, u(\tau)) \, d\tau \, ds
\]

\[
\geq \frac{3\rho}{16} \csc(\theta) \|\Phi u\| = \sigma \|\Phi u\|
\]

Thus, \( \Phi(K) \subset K \)

It is clear that \( \Phi : K \to K \) is a completely continuous mapping.

**Lemma 2.4:** Fixed Point Theorem

Let \( E \) be a Banach space, and let \( K \subset E \) be a cone in \( E \). Assume \( \Omega_1, \Omega_2 \) are open subsets of \( E \) with \( 0 \in \Omega_1 \), \( \bar{\Omega}_1 \subset \Omega_2 \) and let \( \Phi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K \) be a completely continuous operator such that either

1. \( \|\Phi u\| \leq \|u\| \), \( u \in K \cap \partial \Omega_1 \), and \( \|\Phi u\| \|u\| \leq \|u\| \), \( u \in K \cap \partial \Omega_2 \), or
2. \( \|\Phi u\| \geq \|u\| \), \( u \in K \cap \partial \Omega_1 \), and \( \|\Phi u\| \|u\| \leq \|u\| \), \( u \in K \cap \partial \Omega_2 \)

Then \( \Phi \) has a fixed point in \( K \cap (\bar{\Omega}_2 \setminus \Omega_1) \).

We will apply the first and second parts of the above Fixed Point Theorem to the super-linear and sub-linear cases.

**RESULTS**

**Theorem 3.1:** Assume that \( (H_1), (H_2) \) hold, then there has \( \hat{\lambda} \in (0, \infty) \), when \( \hat{\lambda} \in (0, \hat{\lambda}_0] \) the problem (1.1) and (1.2) has at least one positive solution.

Remark \( \hat{m} = \frac{\sin(\theta)}{\sin \theta} \int_0^1 G_i(s,s) \, ds \)

**Proof:** Since \( (H_2) \), we may choose \( \epsilon > 0 \) so that \( f(t,u) \leq \epsilon u \), for \( 0 \leq u \leq r \), where \( \epsilon > 0 \) satisfies

\[
\epsilon \int_0^1 \int_0^1 G_i(s,s)G_2(s,\tau) \, d\tau \, ds \leq \frac{1}{2}
\]

choose \( \hat{\lambda}m \leq \frac{1}{2} \), when \( \hat{\lambda} \in (0, \hat{\lambda}_0] \), let \( \Omega_{0} = \{ u \in C[0,1]; \|u\| < r \} \), \( \forall u \in K \cap \partial \Omega_{0} \), from lemma 2.2, we have

\[
\|\Phi u(t)\| \leq \hat{\lambda}m + \int_0^1 \int_0^1 G_i(s,s)G_2(s,\tau) f(\tau, u(\tau)) \, d\tau \, ds
\]
\[ \leq \lambda \| m + \varepsilon \| \int_0^1 \int_0^1 G_1(s,s)G_2(s,\tau)d\tau ds \]

\[ \leq \lambda \| m + \frac{1}{2} \| u \| \leq \| u \| \]

Then shows \( \| \Phi u \| \leq \| u \| \).

Further, since \( (H_2) \) there exists \( R_1 > 0 \) such that \( f(t,u) \geq \mu u \), \( u \geq R_1 \) where \( \mu > 0 \) chosen so that

\[ \mu \sigma \int_0^1 \int_0^1 G_1(s,s)G_2(s,\tau)d\tau ds \geq 1 \]

Let \( R > \max \{ r, \frac{R_1}{\sigma} \} \) and \( \Omega_2 = \{ u \in [0,1]; \| u \| < R \} \), then \( \forall u \in K \cap \partial \Omega_2 \) and

\[ \min_{t \in [1,3/4]} u(t) \geq \sigma \| u \| = \sigma R > R_1 \), implies

\[ \| \Phi u(t) \| \geq \max_{t \in [0,1]} \text{csch} \left( \rho \right) \rho t (1-t) \int_0^1 \int_0^1 G_1(s,s)G_2(s,\tau)f(\tau, u(\tau))d\tau ds \]

\[ \geq \frac{3\rho}{16} \text{csch} \left( \rho \right) \int_0^1 \int_0^1 G_1(s,s)G_2(s,\tau)f(\tau, u(\tau))d\tau ds \]

\[ \geq \sigma \mu \int_0^1 \int_0^1 G_1(s,s)G_2(s,\tau)d\tau ds \geq \| u \| \]

Hence, \( \| \Phi u \| \geq \| u \| \) for \( \forall u \in K \cap \partial \Omega_2 \).

Therefore, by the first part of the Fixed Point Theorem, it follows that \( \Phi \) has a fixed point in \( K \cap (\Omega_2 \setminus \Omega_1) \).

Further, since \( G_1(t,s)G_2(s,\tau)d\tau ds \geq 0 \), it follows that \( u(t) > 0 \) for \( 0 < t < 1 \).

**Theorem 3.2:** Assume that \( (H_1), (H_2) \) hold, then the Problem \((1.1) \) and \((1.2) \) has at least one positive solution.

**Proof:** Since \((H_3) \), we first choose \( r > 0 \) such that \( f(t,u) \geq \mu u \), for \( 0 \leq u \leq r \) where \( \mu > 0 \) satisfies

\[ \mu \sigma \int_0^1 \int_0^1 G_1(s,s)G_2(s,\tau)d\tau ds \geq 1 \].

Let \( \Omega_1 = \{ u \in C[0,1]; \| u \| < r \} \), for \( \forall u \in K \cap \partial \Omega_1 \), from lemma 2.2, we have

\[ \| \Phi u(t) \| \geq \min_{t \in [0,1]} \text{csch} \left( \rho \right) \rho t (1-t) \int_0^1 \int_0^1 G_1(s,s)G_2(s,\tau)f(\tau, u(\tau))d\tau ds \]

\[ \geq \sigma \int_0^1 \int_0^1 G_1(s,s)G_2(s,\tau)f(\tau, u(\tau))d\tau ds \]

\[ \geq \mu \sigma \| u \| \int_0^1 \int_0^1 G_1(s,s)G_2(s,\tau)d\tau ds \geq \| u \| \]

So that \( \| \Phi u \| \geq \| u \| \)

Now since \((H_3) \), there exists \( H > 0 \) so that \( f(t,u) \leq \varepsilon u \), for \( u \geq H \) where \( \varepsilon > 0 \) satisfies

\[ \varepsilon \int_0^1 \int_0^1 G_1(s,s)G_2(s,\tau)d\tau ds < \frac{1}{2} \]
choose $\lambda, m \leq \frac{1}{2} R$, then when $\lambda \in (0, \lambda_0]$

We consider two cases:

Suppose $f(t, u)$ is unbounded for $\forall 0 < u \leq R$, we have $f(u) \leq f(R), R > \max \{r, H\}$.

Let $\Omega_2 = \{u \in C[0,1]; \|u\| < R\}$, for $\forall u \in K \cap \partial \Omega_2$, therefore

$$\|\Phi u(t)\| \leq \lambda m + \int_0^1 \int_0^1 G_1(s, s)G_2(s, \tau)f(\tau, u(\tau))d\tau ds$$

$$\leq \lambda m + \int_0^1 \int_0^1 G_1(s, s)G_2(s, \tau)f(\tau, u(\tau))d\tau ds$$

$$\leq \lambda m + \frac{1}{2} R < R = \|u\|$$

So that $\|\Phi u\| \leq \|u\|$.

Suppose $f(t, u)$ is bounded, there exists $N > 0$, for $t \in [0,1]$ and $u \in [0, +\infty)$ we have $f(t, u) \leq N, R > \max \{r, 2N\} \int_0^1 \int_0^1 G_1(s, s)G_2(s, \tau)d\tau ds$. Let $\Omega_2 = \{u \in C[0,1]; \|u\| < R\}$, for $\forall u \in K \cap \partial \Omega_2$, from Lemma 2.2, we have

$$\|\Phi u(t)\| \leq \lambda m + N \int_0^1 \int_0^1 G_1(s, s)G_2(s, \tau)f(\tau, u(\tau))d\tau ds$$

$$\leq \lambda m + \frac{1}{2} R < R = \|u\|$$

So that $\|\Phi u\| \leq \|u\|$.

Therefore, in either case we may put $\Omega_2 = \{u \in [0,1]; \|u\| < R\}$ and for $\forall u \in K \cap \partial \Omega_2$ we have $\|\Phi u\| \leq \|u\|$. By the second part of the Fixed Point Theorem it follows that BVP (1.1), (1.2) has a positive solution, and this completes the proof of the theorem.

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