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Research Article

Existence of Positive Solutions of Nonlinear Fourth-order Boundary Problem with Parameter

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Abstract: This paper is concerned with the fourth-order boundary problem

$$\begin{cases} u^{(4)}(t)-\rho^4 u(t)=f(t,u(t)) \\ u(0)=0,u(1)=0 \\ u''(0)=0,u''(1)=\lambda \end{cases}$$

where and .Combine with the properties of the Green function using Fixed Point theorem in cones, proved the existence of positive solutions nonlinear fourth-order boundary value problem
Keywords: Fourth-order Boundary value problem, one, Positive solutions, Fixed point

MSC: 34B10, 34B15

INTRODUCTION

In this paper , we think of the nonlinear fourth-order boundary value problems (BVP for short)

$$u^{(4)}(t)-\rho^4 u(t)=f(t,u(t)),0<t<1, \tag{1.1}$$

$$u(0)=0,u(1)=0,u''(0)=0,u''(1)=\lambda \tag{1.2}$$

where $\lambda>0$ and $0<\rho<\frac{\pi}{2}$ is a parameter , $f:[0,1]\times[0,+\infty)\to R$ is a nonnegative and continuous function. Function $u(t)$ which is positive on $(0,1)$ and $u(t)\in C^3[0,1]\cap C^4[0,1]$, if $u(t)$ satisfied differential equation (1.1) and the boundary conditions (1.2) ,we call it is the positive solution of the nonlinear fourth-order boundary problem of (1.1) .It is assumed throughout that

(H_1): $f(t,u)$ is integral for each fixed $u\in[0,1]\times[0,+\infty)$, and $0<\int_0^1 f(t,u(t))\mathrm{d}t<+\infty$;

(H_2): $\lim\limits_{u\to 0^+}\sup\limits_{t\in[0,1]}\frac{f(t,u)}{u}=0,\lim\limits_{u\to\infty}\inf\limits_{t\in[0,1]}\frac{f(t,u)}{u}=\infty$;

(H_3): $\lim\limits_{u\to 0^+}\sup\limits_{t\in[0,1]}\frac{f(t,u)}{u}=\infty,\lim\limits_{u\to\infty}\inf\limits_{t\in[0,1]}\frac{f(t,u)}{u}=0.$

It is well-known that the fixed point theorem method is a powerful tool for proving the existence results for boundary value problem (BVP for short). It has been used to deal with the multi-point BVP for second-order ordinary differential equations and the two-point BVP for higher-order ordinary differential equations, see [1-4]. But there are fewer results on multi-point higher-order BVPs in the literature. In 2006, by using the upper and lower solution method, the authors studied the following fourth-order four-point BVP[5].

$$\begin{cases} u^{(4)}(t) = f\left(t,u(t),u''(t)\right), t \in [0,1] = I \\ u(0) = 0, u(1) = 0 \\ au''(\xi_1) - bu'''(\xi_1) = 0, \mathrm{c}u''(\xi_2) + du'''(\xi_2) = 0, \end{cases} \quad (1.3)$$

They obtained the existence results for BVP under the condition $f(t,u,v)$ is increasing on u and decreasing on v, i.e

$$\begin{aligned} & f(t,u_2,v) - f(t,u_1,v) \ge 0, u_1 \le u_2 \\ & f(t,u,v_2) - f(t,u,v_1) \le 0, \mathrm{v}_1 \le v_2 \end{aligned} \quad (1.4)$$

De-Xiang Ma and Xiao-Zhong Yang [4] by using the upper and lower solution method, proved the fourth-order four-point boundary value problem

[5] Where, $\eta, \xi \in (0,1)$ and $a, b \ge 0$ .They release the conditions imposed on $f(t,u,v)$ from (1.4) to

$$\begin{aligned} & f(t,u_2,v) - f(t,u_1,v) \ge -\lambda_1(u_2 - u_1), u_1 \le u_2 \\ & f(t,u,v_2) - f(t,u,v_1) \le \lambda_2(v_2 - v_1), \mathrm{v}_1 \le v_2 \end{aligned} \quad (1.5)$$

Where, λ_1 and λ_2 are two nonnegative numbers. $f(t,u,v)$ is weak-increasing on u and weak-decreasing on v .They gave a critical theorem, a new maximum principle. Inspired and motivated by the works mentioned, we study a group of contains parameter of nonlinear fourth-order boundary value problems, proved the existence of positive solution.

Preliminary

In this section, we will give some preliminary considerations and some lemmas which are essential to our main result.

Lemma 2.1: Assume m,n,q are constants, $\varphi_1(t), \varphi_2(t)$ are two independent solutions of the non-homogeneous equation $mv''(t) + nv'(t) + qv(t) = h(t)$, $\varphi_0(t)$ is one of the solutions of the boundary problem (2.1),from the general solution of non-homogeneous equation, we can get $\varphi(t) = c_1\varphi_1(t) + c_2\varphi_2(t) + \varphi_0(t)$ is the general solution of the equation $av''(t) + bv'(t) + cv(t) = h(t)$, where c_1, c_2 are any two constants.

$$\begin{cases} mv''(t) + nv'(t) + qv(t) = h(t) \in L^1(0,1), \\ v(0) = 0, v(1) = 0. \end{cases} \quad (2.1)$$

Proof: It can be validation directly by the structure of non-singular equation.

Consider the nonlinear second order boundary problem first.

$$\begin{cases} u''(t) - \rho^2 u(t) = -v(t) \\ u(0) = 0, u(1) = 0 \end{cases} \quad (2.2)$$

It is easily to compute (2.2) is equivalent to the following integral equation

$$u(t) = \int_0^1 G_1(t,s) v(s)\,\mathrm{d}s, \quad (2.3)$$

Where

$$G_1(t,s) = \begin{cases} \dfrac{\sinh(\rho s)\sinh(\rho - \rho t)}{\rho \sinh(\rho)}, 0 \le s \le t \le 1 \\ \dfrac{\sinh(\rho t)\sinh(\rho - \rho s)}{\rho \sinh(\rho)}, 0 \le t \le s \le 1 \end{cases} \quad (2.4)$$

Consider the nonlinear second order boundary problem

$$\begin{cases} v''(t)+\rho^2 v(t)=-f\left(t,u(t)\right) \\ v(0)=0, v(1)=\lambda \end{cases} \tag{2.5}$$

we have already know the nonlinear second order boundary problem

$$\begin{cases} v''(t)+\rho^2 v(t)=-f\left(t,u(t)\right) \\ v(0)=0, v(1)=0 \end{cases}$$

is equivalent to the following integral equation

$$v(t)=\int_0^1 G_2(t,s)\, f\left(s,u(s)\right)\mathrm{ds},$$

where

$$G_2(t,s)=\begin{cases} \dfrac{\sin\rho s\sin\rho(1-t)}{\rho\sin\rho}, 0\le s\le t\le 1 \\ \dfrac{\sin\rho t\sin\rho(1-s)}{\rho\sin\rho}, 0\le t\le s\le 1 \end{cases} \tag{2.6}$$

And because of $\varphi_1(t)=\cos(\rho t), \varphi_2(t)=\sin(\rho t)$ are two independent solutions of equation $v''(t)+\rho^2 v(t)=0$,from lemma 2.1,we can say the general solution of boundary problem (2.5) can be represented $v(t)=c_1\cos(\rho t)+c_2\sin(\rho t)+\int_0^1 G_2(t,s)\, f(s)\mathrm{ds}$,also satisfy the conditions $v(0)=0, \mathrm{v}(1)=\lambda$, according to this boundary condition we can calculate the coefficient of c_1, c_2 ,after computing and tiding ,the existence of boundary problem (2.5) can use the following integral equation

$$v(t)=\frac{\lambda\sin\rho t}{\sin\rho}+\int_0^1 G_2(t,s)\, f\left(s,u(s)\right)\mathrm{ds} \tag{2.7}$$

put (2.7) into (2.3) ,we receive the solution of the nonlinear boundary problem

$$u(t)=\frac{\lambda\sin(\rho s)}{\sin\rho}\int_0^1 G_1(t,s)\mathrm{ds}+\int_0^1\int_0^1 G_1(t,s) G_2(s,\tau)\, f\left(\tau,u(\tau)\right)\mathrm{d}\tau\mathrm{ds}$$

Lemma 2.2: Foe all $(s,t)\in[0,1]\times[0,1]$, we have

$$\frac{G_1(t,s)}{G_1(s,s)}=\begin{cases} \dfrac{\sinh(\rho-\rho t)}{\sinh(\rho-\rho s)}, 0\le s\le t\le 1 \\ \dfrac{\sinh(\rho t)}{\sinh(\rho s)}, 0\le t\le s\le 1 \end{cases}$$

$$\rho t(1-t)\operatorname{csch}(\rho) G_1(s,s)\le G_1(t,s)\le G_1(s,s)$$

Proof: It is clearly to see

$$\frac{G_1(t,s)}{G_1(s,s)}=\begin{cases} \dfrac{\sinh(\rho-\rho t)}{\sinh(\rho-\rho s)}, 0\le s\le t\le 1 \\ \dfrac{\sinh(\rho t)}{\sinh(\rho s)}, 0\le t\le s\le 1 \end{cases}$$

$$\ge\begin{cases} \rho t(1-t)\operatorname{csch}(\rho), 0\le s\le t\le 1 \\ \rho t\operatorname{csch}(\rho), 0\le t\le s\le 1 \end{cases}$$

$$\ge\rho t(1-t)\operatorname{csch}(\rho)$$

It is obvious that $G_1(t,s) \le G_1(s,s)$.The proof is complete.

Define an integral operator $\Phi : C^+[0,1] \to C^+[0,1]$ by

$$\Phi u(t) = \frac{\lambda \sin(\rho s)}{\sin \rho}\int_0^1 G_1(t,s)\mathrm{d}s + \int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)\, f(\tau,u(\tau))\,\mathrm{d}\tau\mathrm{d}s \qquad (2.8)$$

Then, only if nonzero fixed point $u(t)$ of mapping Φ defined by (2.8) is a positive solution of (1.1) and (1.2)

Lemma 2.3: $\Phi(K) \subset K$

Proof: For any $u \in K$, from lemma 2.2 we have

$$\|\Phi u(t)\| = \max \frac{\lambda \sin(\rho s)}{\sin \rho}\int_0^1 G_1(t,s)\mathrm{d}s + \int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)\, f(\tau,u(\tau))\,\mathrm{d}\tau\mathrm{d}s$$

And inequalities

$$\|\Phi u(t)\| \le \max \frac{\lambda \sin(\rho s)}{\sin \rho}\int_0^1 G_1(s,s)\mathrm{d}s + \int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)\, f(\tau,u(\tau))\,\mathrm{d}\tau\mathrm{d}s$$

$$\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} \Phi u(t) \ge \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} \frac{2\rho t(1-t)}{e^{\rho}-e^{-\rho}}\left[\frac{\lambda \sin(\rho s)}{\sin \rho}\int_0^1 G_1(s,s)\mathrm{d}s + \int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)\, f(\tau,u(\tau))\,\mathrm{d}\tau\mathrm{d}s\right]$$

$$\ge \frac{3\rho}{16}\operatorname{csch}(\rho)\|\Phi u\|$$

$$= \sigma \|\Phi u\|$$

Thus, $\Phi(k) \subset K$

It is clear that $\Phi : K \to K$ is a completely continuous mapping.

Lemma 2.4: Fixed Point Theorem

Let E be a Banach space, and let $K \subset E$ be a cone in E .Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$ and let $\Phi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either

(1) $\|\Phi u\| \le \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|\Phi u\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$;or

(2) $\|\Phi u\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$, and $\|\Phi u\| \le \|u\|$, $u \in K \cap \partial\Omega_2$

Then Φ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

We will apply the first and second parts of the above Fixed Point Theorem to the super-linear and sub-linear cases.

RESULTS

Theorem 3.1: Assume that $(H_1),(H_2)$ hold, then there has $\lambda_0 \in (0,\infty)$,when $\lambda \in (0,\lambda_0]$ the problem (1.1) and (1.2) has at least one positive solution.

Remark $m = \dfrac{\sin(\rho s)}{\sin \rho}\displaystyle\int_0^1 G_1(s,s)\mathrm{d}s$

Proof: Since (H_2), we may choose $r > 0$ so that $f(t,u) \le \varepsilon u$,for $0 \le u \le r$, where $\varepsilon > 0$ satisfies

$$\varepsilon\int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s \le \frac{1}{2},$$

choose $\lambda_0 m \le \frac{1}{2} r$,when $\lambda \in (0,\lambda_0]$,let $\Omega_1 = \{u \in C[0,1]; \|u\| < r\}$ $\forall u \in K \cap \partial\Omega_1$ from lemma 2.2,we have

$$\|\Phi u(t)\| \le \lambda m + \int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)\, f(\tau,u(\tau))\,\mathrm{d}\tau\mathrm{d}s$$

$$\leq \lambda_0 m + \varepsilon \|u\| \int_0^1 \int_0^1 G_1(s,s) G_2(s,\tau) \mathrm{d}\tau \mathrm{d}s$$

$$\leq \lambda_0 m + \frac{1}{2}\|u\| \leq \|u\|$$

Then shows $\|\Phi u\| \leq \|u\|$.

Further, since (H_2) there exists $R_1 > 0$ such that $f(t,u) \geq \mu u$, $u \geq R_1$ where $\mu > 0$ chosen so that

$$\mu\sigma \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,s) G_2(s,\tau) \mathrm{d}\tau \mathrm{d}s \geq 1$$

Let $R > \max\{r, \frac{R_1}{\sigma}\}$ and $\Omega_2 = \{u \in C[0,1]; \|u\| < R\}$, then $\forall u \in K \bigcap \partial\Omega_2$ and $\min_{t\in[1/4,3/4]} u(t) \geq \sigma \|u\| = \sigma R > R_1$, implies

$$\|\Phi u(t)\| \geq \max_{t\in[0,1]} \operatorname{csch}(\rho) \rho t(1-t) \int_0^1 \int_0^1 G_1(s,s) G_2(s,\tau) f(\tau, u(\tau)) \mathrm{d}\tau \mathrm{d}s$$

$$\geq \frac{3\rho}{16} \operatorname{csch}(\rho) \int_0^1 \int_0^1 G_1(s,s) G_2(s,\tau) f(\tau, u(\tau)) \mathrm{d}\tau \mathrm{d}s$$

$$\geq \sigma\mu \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,s) G_2(s,\tau) u(\tau) \mathrm{d}\tau \mathrm{d}s$$

$$\geq \sigma\mu \|u\| \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,s) G_2(s,\tau) \mathrm{d}\tau \mathrm{d}s \geq \|u\|$$

Hence, $\|\Phi u\| \geq \|u\|$ for $\forall u \in K \bigcap \partial\Omega_2$

Therefore, by the first part of the Fixed Point Theorem, it follows that Φ has a fixed point in $K \bigcap (\bar{\Omega}_2 \setminus \Omega_1)$ .Further, since $G_1(t,s) G_2(s,\tau) \mathrm{d}\tau \mathrm{d}s \geq 0$, it follows that $u(t) > 0$ for $0 < t < 1$.

Theorem 3.2: Assume that $(H_1),(H_3)$ hold, then the Problem (1.1) and (1.2) has at least one positive solution.

Proof: Since (H_3), we first choose $r > 0$ such that $f(t,u) \geq \mu u$, for $0 \leq u \leq r$ where $\mu > 0$ satisfies

$$\mu\sigma \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,s) G_2(s,\tau) \mathrm{d}\tau \mathrm{d}s \geq 1,$$

Let $\Omega_1 = \{u \in C[0,1]; \|u\| < r\}$, for $\forall u \in K \bigcap \partial\Omega_1$, from lemma 2.2, we have

$$\|\Phi u(t)\| \geq \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} \operatorname{csch}(\rho) \rho t(1-t) \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,s) G_2(s,\tau) f(\tau, u(\tau)) \mathrm{d}\tau \mathrm{d}s$$

$$\geq \sigma \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,s) G_2(s,\tau) f(\tau, u(\tau)) \mathrm{d}\tau \mathrm{d}s$$

$$\geq \mu\sigma \|u\| \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(s,s) G_2(s,\tau) \mathrm{d}\tau \mathrm{d}s \geq \|u\|$$

So that $\|\Phi u\| \geq \|u\|$

Now since (H_3), there exists $H > 0$ so that $f(t,u) \leq \varepsilon u$, for $u \geq H$ where $\varepsilon > 0$ satisfies

$$\varepsilon \int_0^1 \int_0^1 G_1(s,s) G_2(s,\tau) \mathrm{d}\tau \mathrm{d}s < \frac{1}{2}$$

choose $\lambda_0 m \le \frac{1}{2}R$,then when $\lambda \in (0, \lambda_0]$

We consider two case:

Suppose $f(t,u)$ is unbounded for $\forall 0 < u \le R$,we have $f(u) \le f(R)$, $R > \max\{r, H\}$, .

Let $\Omega_2 = \{u \in C[0,1]; \|u\| < R\}$, for $\forall u \in K \cap \partial\Omega_2$ therefore

$$\|\Phi u(t)\| \le \lambda m + \int_0^1\int_0^1 G_1(s,s)G_2(s,\tau) f(\tau, u(\tau))\mathrm{d}\tau\mathrm{d}s$$

$$\le \lambda_0 m + \int_0^1\int_0^1 G_1(s,s)G_2(s,\tau) f(R)\mathrm{d}\tau\mathrm{d}s$$

$$\le \lambda_0 m + \varepsilon R\int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s$$

$$\le \lambda_0 m + \frac{1}{2}R < R = \|u\|$$

So that $\|\Phi u\| \le \|u\|$.

Suppose $f(t,u)$ is bounded , there exists $N > 0$, for $t \in [0,1]$ and $u \in [0, +\infty)$ we have $f(t,u) \le N$, $R > \max\{r, 2N\int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s\}$,Let $\Omega_2 = \{u \in C[0,1]; \|u\| < R\}$,for $\forall u \in K \cap \partial\Omega_2$, from lemma 2.2, we have

$$\|\Phi u(t)\| \le \lambda m + \int_0^1\int_0^1 G_1(s,s)G_2(s,\tau) f(\tau, u(\tau))\mathrm{d}\tau\mathrm{d}s$$

$$\le \lambda_0 m + N\int_0^1\int_0^1 G_1(s,s)G_2(s,\tau)\mathrm{d}\tau\mathrm{d}s$$

$$\le \lambda_0 m + \frac{1}{2}R \le R = \|u\|$$

So that $\|\Phi u\| \le \|u\|$.

Therefore, in either case we may put $\Omega_2 = \{u \in C[0,1]; \|u\| < R\}$ and for $\forall u \in K \cap \partial\Omega_2$ we have $\|\Phi u\| \le \|u\|$.By the second part of the Fixed Point Theorem it follows that BVP$(1.1),(1.2)$ has a positive solution, and this completes the proof of the theorem.

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