Research Article

On Special Diophantine Triples
M. A. Gopalan¹, S. Vidhyalakshmi²*, N. Thiruniraiselvi³

¹,² Professor, Department of Mathematics, SIGC, Trichy-620002, Tamilnadu, India
³ Research Scholar, Department of Mathematics, SIGC, Trichy-620002, Tamilnadu, India

*Corresponding author
S. Vidhyalakshmi
Email: vidhyasgc@gmail.com

Abstract: This paper concerns with the study of constructing a special Diophantine triples (a,b,p) such that the product of any two elements of the set added with their sum and increased by a polynomial with integer coefficients is a Perfect square.

Keywords: Diophantine Triples, Pell equation

2010 Mathematics Subject Classification: 11D99

INTRODUCTION

A Set of positive integers (a₁,a₂,a₃,......aₘ) is said to have the property D(n), if aᵢaⱼ + n is a perfect square for all 1 ≤ i < j ≤ m and such a set is called a Diophantine m-tuple with property D(n). Many mathematicians considered the problem of the existence of Diophantine quadruples with the property D(n) for any arbitrary integer n [1] and also for any linear polynomial in n. Further, various authors considered the connections of the problems of diophantus, davenport and Fibonacci numbers in [2-19]. In this communication, we find special Diophantine triple (k, k+3, 4k+8) in which the product of any two elements of the set added with their sum and increased by (1-k) is a perfect square.

Construction Of Special Diophantine Triple With Property D(1-K)

Let a = k, b = k + 3 be any two non-zero distinct integers such that ab + a + b + (1-k) is a perfect square. We search for a non-zero distinct integer p such that in the triple (a,b,p) the product of any two elements added with their sum and increased by (1-k) is a perfect square.

That is,

\[ p(k+1)+1 = \alpha^2 \]  (1)
\[ p(k+4)+4 = \beta^2 \]  (2)

Eliminating “p” we get

\[ (k+1)\beta^2 - (k+4)\alpha^2 = 3k \]  (3)

The introduction of the linear transformations

\[ \alpha = X + (k+1)T \]  (4)
\[ \beta = X + (k+4)T \]  (5)

In (3) leads to the pell equation

\[ X^2 = (k^2 +3k+2)T^2 + (k+2) \]  (6)

Whose initial solution is \( T₀ = 1, X₀ = k + 2 \). Thus (4) yields \( \alpha₀ = 2k + 3 \) and using (1), we get \( p = 4k + 8 \)
Hence \((a,b,p) = (k, k+3, 4k+8)\) is the required special Diophantine triples with property \(D(1-k)\).

The repetition of the above process leads to the generation of special Diophantine triples

\[
(F^2_{m+3}k + F^2_{m+4}k + F^2_{m+5}k + F^2_{m+6}k + 1) \quad \text{with property} \quad D(1-k).
\]

Here \(F_{-1} = 1, F_0 = 1, F_1 = 2, F_2 = 3,...\)

For illustration, a few examples are presented below

\((k + 3, 4k + 8, 9k + 24), (4k + 8, 9k + 24, 25k + 63), (9k + 24, 25k + 63, 64k + 168),
\(25k + 63, 64k + 168, 169k + 440), (64k + 168, 169k + 440, 441k + 1155)\)

**REMARK 1**

Note that, when \(k=0\), the triple \((0, 3, 8)\) is the special Diophantine triple with property \(D(1)\). Observe that

\[3 = 2^2 - 1, 8 = 3^2 - 1,\] if \(\alpha_0\) is any non-zero integer such that \((0, 3, 8, \alpha_0)\) is the special Diophantine quadruples with property \(D(1)\), then it is seen that \(\alpha_0 = 24 = 5^2 - 1\).

The repetition of the above process leads to the generation of special Diophantine tuples

\(\{0, F^2_1 - 1, F^2_2 - 1, F^2_3 - 1,..., F^2_{m+2} - 1,\}\) with property \(D(1)\).

Where \(F_{-1} = 1, F_0 = 1, F_{m+2} = F_m + F_{m+1}, (m = -1, 0, 1, 2, 3, \ldots)\)

**REMARK 2:**

Replacing \(k\) by a Gaussian integer and irrational numbers respectively in each of the above triples, it is noted that each resulting triple is a Gaussian triple and irrational triple satisfying the required property.

<table>
<thead>
<tr>
<th>(k)</th>
<th>Triples ((a, b, p))</th>
<th>Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1 + i\sqrt{3})</td>
<td>((4 + i\sqrt{3}, 12 + i4\sqrt{3}, 33 + i9\sqrt{3}), (12 + i4\sqrt{3}, 33 + i9\sqrt{3}, 88 + i25\sqrt{3}), (33 + i9\sqrt{3}, 88 + i25\sqrt{3}, 232 + i64\sqrt{3}))</td>
<td>(D(-i\sqrt{3}))</td>
</tr>
<tr>
<td>(2 + i3)</td>
<td>((5 + i3, 16 + i12, 27), (16 + i12, 27, 113 + i75), (113 + i75, 296 + i192, 192, 778, 192, 1323))</td>
<td>(D(-1 - i3))</td>
</tr>
</tbody>
</table>

**CONCLUSION**

To conclude, one may search for other choices of triples with suitable property.

**Acknowledgement**

*The financial support from the UCG, New Delhi (F-MRP-5123/14(SERO/UCG) dated March 2014) for a part of this work is gratefully acknowledged.*

**REFERENCES**

3. Brown E; Sets in which \(xy + k\) is always a square. Math Comp., 1985; 45: 613-620.
13. Fujita Y, Togbe A; Uniqueness of the extension of the $D(4k^2)$ triple $(k^2 - 4, k^2, 4k^2 - 4)$. NNTDM 17, 2011; 4: 42-49.
18. Fujita Y; The unique representation $d = 4k(k^2 - 1)$ in D(4)-quadruples{$k-2,k+2,4k,d$}. Math Commun., 2006; 11: 69-81.