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Research Article

Sufficient conditions for certain analytic functions

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Abstract: Let f(z) be an analytic function in the open unit disk U normalized with $f(0)=0$ and $f'(0)=1$. In this paper, we deal with the question of another conditions different from [6]. This is the extension of their works.
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INTRODUCTION

Let H denote the class of analytic functions in $U=\{z\in C:|z|<1\}$, and A denote the subclass of H , which consist as functions of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots U \qquad (1)$$

A function $f(z)\in A$ is consist as starlike of order $\alpha(0\le\alpha<p)$ in U (see [1]), that is, $f(z)\in S^*(\alpha)$, if and only if

$$Re(\frac{zf'(z)}{f(z)})>\alpha, 0\le\alpha<1, z\in U \qquad (2)$$

with $S^*(0):=S^*$.

Similarly, a function $f(z)\in A$ is consist as convex of order $\alpha(0\le\alpha<1)$ in U (see [1]), that is, $f(z)\in K(\alpha)$, if and only if

$$Re(1+\frac{zf''(z)}{f'(z)})>\alpha, 0\le\alpha<1, z\in U \qquad (3)$$

with $K(0)=K$.

According to the definitions for the classes $S^*(\alpha)$ and $K(\alpha)$, we know that $f(z)\in K(\alpha)$ if and only if $zf'(z)\in S^*(\alpha)$. Marx [2] and Strohhäcker [3] showed that $f(z)\in K(0)$ implies $f(z)\in S^*(1/2)$.

Ozaki [4] and Kaplan [5] investigated the following functions : If $f\in A$ satisfies

$$Re(\frac{f'(z)}{g'(z)})>0, z\in U \qquad (4)$$

for some convex function $g(z)$, then $f(z)$ is univalent function in U . In the view of Kaplan (see [5]), we say that $f(z)$ satisfying the above inequality is close-to-convex in U , that is, $f(z)\in C(0):=C$.

It is well known that the above definition concerning close-to-convex functions, is equivalent to the following condition:

$$Re(\frac{zf'(z)}{g(z)})>0, z\in U \qquad (5)$$

for some starlike function $g(z) \in A$.

A function $f(z) \in A$ is consist as close-to-convex of order $\alpha(0 \le \alpha < p)$ in U with respect to $g(z)$, that is, $f(z) \in C(\alpha)$, if and only if

$$Re(\frac{zf'(z)}{g(z)}) > \alpha, z \in U \tag{6}$$

for some real $\alpha(0 \le \alpha < 1)$ and for some starlike function $g(z) \in A$.

Nunokawa et al. investigated the order α of close-to-convex functions (see [6]). In this paper, we deal with the question of another conditions different from [6]. This is the extension of their works.

MAIN RESULTS

Lemma 2.1. (see [7])*Let* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *be analytic in the unit disc* U *and suppose that there exists a point* $z_0 \in U$ *such that*

$$Rep(z) > 0 \; for \; |z| < |z_0| \tag{7}$$

and

$$Rep(z_0) = 0. \tag{8}$$

Then we have

$$z_0 p'(z_0) \le -\frac{1}{2}(1 + |p(z_0)|^2). \tag{9}$$

Making use of Lemma 2.1, we first prove the following Theorem.

Theorem 2.1. *Let* $f(z) \in A$*, and suppose that there exists a starlike function* $g(z)$ *such that*

$$Re\{\frac{zf'(z)}{g(z)}(1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)})\} > -\frac{1}{2}(1 + |\frac{zf'(z)}{g(z)}|^2), z \in U \tag{10}$$

then $f(z) \in C$.

Proof. Let us put

$$p(z) = \frac{zf'(z)}{g(z)}, \tag{11}$$

then $p(z)$ is analytic in U and $p(0) = 1$. Suppose that there exists a point $z_0 \in U$ which satisfies the conditions (7) and (8) of Lemma 2.1.

Making use of (11), it follows that

$$\frac{z_0 f'(z_0)}{g(z_0)}(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 g'(z_0)}{g(z_0)}) = z_0 p'(z_0). \tag{12}$$

Since the function $p(z)$ and the point z_0 satisfy all conditions Lemma 2.1, therefore in view of (9), we obtain

$$\begin{aligned} Re\{\frac{z_0 f'(z_0)}{g(z_0)}(1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 g'(z_0)}{g(z_0)})\} &\le -\frac{1}{2}(1 + |p(z_0)|^2) \\ &= -\frac{1}{2}(1 + |\frac{z_0 f'(z_0)}{g(z_0)}|^2). \end{aligned} \tag{13}$$

This is a contradiction with (10) and therefore proof of the Theorem 2.1 is completed.

Theorem 2.2. *Let* $f(z) \in A$*, and suppose that there exists a starlike function* $g(z)$ *such that*

$$Re\{\frac{zf'(z)}{g(z)}(1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)})\}>-\frac{1}{4}(1+|\frac{zf'(z)}{g(z)}|^2), z\in U \tag{14}$$

then $f(z)\in C(1/2)$.

Proof. Let us put

$$p(z)=2(\frac{zf'(z)}{g(z)}-\frac{1}{2}), \tag{15}$$

then $p(z)$ is analytic in U and $p(0)=1$. Suppose that there exists a point $z_0\in U$ which satisfies the conditions (7) and (8) of Lemma 2.1.

Now using (15), it follows that

$$\frac{z_0f'(z_0)}{g(z_0)}(1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0g'(z_0)}{g(z_0)})=\frac{1}{2}z_0p'(z_0). \tag{16}$$

Since the function $p(z)$ and the point z_0 satisfy all conditions Lemma 2.1, therefore in view of (9), we obtain

$$Re\{\frac{z_0f'(z_0)}{g(z_0)}(1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0g'(z_0)}{g(z_0)})\}\le-\frac{1}{4}(1+|p(z_0)|^2)$$
$$=-\frac{1}{4}(1+|\frac{z_0f'(z_0)}{g(z_0)}|^2). \tag{17}$$

This is a contradiction with (14) and therefore proof of the Theorem 2.2 is completed.

Theorem 2.3. *Let* $f(z)\in A, 0\le\alpha<1$ *and suppose that there exists a starlike function* $g(z)$ *such that*

$$Re\{\frac{zf'(z)}{g(z)}(1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)})\}>-\frac{1}{2}(1-\alpha), z\in U \tag{18}$$

then $f(z)\in C(\alpha)$.

Proof. Let us put

$$\frac{zf'(z)}{g(z)}=(1-\alpha)p(z)+\alpha, \tag{19}$$

then $p(z)$ is analytic in U and $p(0)=1$. Suppose that there exists a point $z_0\in U$ which satisfies the conditions (7) and (8) of Lemma 2.1.

Making use of (19), it follows that

$$\frac{z_0f'(z_0)}{g(z_0)}(1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0g'(z_0)}{g(z_0)})=(1-\alpha)z_0p'(z_0). \tag{20}$$

Since the function $p(z)$ and the point z_0 satisfy all conditions Lemma 2.1, therefore in view of (9), we obtain

$$Re\{\frac{z_0f'(z_0)}{g(z_0)}(1+\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0g'(z_0)}{g(z_0)})\}\le-\frac{1}{2}(1-\alpha)(1+|p(z_0)|^2)$$
$$\le-\frac{1}{2}(1-\alpha). \tag{21}$$

This is a contradiction with (18) and therefore proof of the Theorem 2.3 is completed.

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