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Research Article

On The Integral Solutions Of The Binary Quadratic Equation $x^2=15y^2-11^t$

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Abstract: The binary quadratic Diophantine equation represented by $x^2 = 15y^2 - 11^t$, t odd is analysed for its non-zero distinct integer solutions. Employing the lemma of Brahmagupta, infinitely many integral solutions of the above Pell equation are obtained. The recurrence relations on the solutions are also presented. A few interesting relations among the solutions are given. Further, there exist no integer solutions when t is even.
Keywords: Binary quadratic, Pell equation., Integer solutions

INTRODUCTION

It is well known that the Pell equation $x^2 - Dy^2 = \pm 1$, (D>0 and square free) has always positive integer solutions. When $N \neq 1$, the Pell equation $x^2 - Dy^2 = N$ may not have any positive integer solutions. For example the equations $x^2 = 3y^2 - 1$ and $x^2 = 7y^2 - 4$ have no positive integer solutions. When k is a positive integer and $D \in \{k^2 \pm 4, k^2 \pm 1\}$, positive integer solutions of the equations $x^2 - Dy^2 = \pm 4$ and $x^2 - Dy^2 = \pm 1$ have been investigated by Jones in [4]. The same or similar equations are investigated in [3,6,9,10]. In [1,2,5,7,8,11,12,13] some specific Pell equation and their integer solutions are considered. In [14], the integer solutions of Pell equation $x^2 - (k^2 + k)y^2 = 2^t$ has been considered. In [15], the Pell equation $x^2 - (k^2 - k)y^2 = 2^t$ is analyzed for the integer solutions.

This communication concerns with the Pell equation $x^2 = 15y^2 - 11^t$ and infinitely many positive integer solutions are obtained when t is odd The recurrence relations on the solutions are also given. Further, it is observed that, when t is even there exist no integer solutions of the considered Pell equation

METHOD OF ANALYSIS:

The Pell equation to be solved is $x^2 = 15y^2 - 11^t, t = 2m+1$ (1)

First, we consider the Pell equation $x^2 = 15y^2 - 11$ (2)

whose fundamental solution is $(\tilde{x}_0, \tilde{y}_0) = (2,1)$.

The other solutions of (2) can be derived from the relations

$$\tilde{x}_n = \frac{f_n}{2}, \qquad \tilde{y}_n = \frac{g_n}{2\sqrt{15}}$$

where
$$f_n = [(2+\sqrt{15})^{n+1} + (2-\sqrt{15})^{n+1}]$$
$$g_n = [(2+\sqrt{15})^{n+1} - (2-\sqrt{15})^{n+1}]$$

Now, we consider the general equation

$$x^2 = 15y^2 - 11^{2m+1}, m \geq 1 \quad (3)$$

The initial solution of (3) is

$$X_1 = 13*11^{m-1}, \quad Y_1 = 10*11^{m-1}$$

Applying the lemma of Bramagupta between (X_1, Y_1) and the solutions of the classical pell equation $x^2 = 15y^2 + 1$, the other solutions of (3) can be obtained from the relations

$$X_{n+2} = 11^{m-1}[\frac{13f_n}{2} + \frac{150g_n}{2\sqrt{15}}]$$

$$Y_{n+2} = 11^{m-1}[\frac{10f_n}{2} + \frac{13g_n}{2\sqrt{15}}]$$

The recurrence relations satisfied by the solution of (1) are found to be

$$X_{n+4} - 8X_{n+3} + X_{n+2} = 0$$

$$Y_{n+4} - 8Y_{n+3} + Y_{n+2} = 0$$

$$X_1 = \begin{cases} 2 & if \quad m = 0 \\ 13*11^{m-1} & if \quad m \geq 1 \end{cases}$$

$$X_2 = \begin{cases} 19 & if \quad m = 0 \\ 202*11^{m-1} & if \quad m \geq 1 \end{cases}$$

and

$$Y_1 = \begin{cases} 1 & if \quad m = 0 \\ 10*11^{m-1} & if \quad m \geq 1 \end{cases}$$

$$Y_2 = \begin{cases} 4 & if \quad m = 0 \\ 53*11^{m-1} & if \quad m \geq 1 \end{cases}$$

A few interesting properties satisfied by the solutions of (1) are exhibited below:

$(i) Y_{n+3} = X_{n+2} + 4Y_{n+2}$

$(ii) X_{n+3} = 4X_{n+2} + 15Y_{n+2}$

$(iii) Y_{n+4} = 8X_{n+2} + 31Y_{n+2}$

$(iv) X_{n+4} = 31X_{n+2} + 120Y_{n+2}$

$(v) Y_{n+4} = X_{n+3} + 4Y_{n+3}$

$(vi) X_{n+4} = 4X_{n+3} + 15Y_{n+3}$

$(vii) X^2_{n+4} - X^2_{n+2} = 240X_{n+3}Y_{n+3}$

Each of the following triples forms an A.P

$(a)(X_{n+2}, 4X_{n+3}, X_{n+4})$

$(b)(X_{n+3}, 2Y_{n+3}, Y_{n+2})$

$(c)(X_{n+2}, 15Y_{n+3}, X_{n+4})$

APPLICATIONS:

1. Define $r = X_{n+2} + \frac{Y_{n+2}}{2}$, $s = \frac{Y_{n+2}}{2}$ where (X_{n+2}, Y_{n+2}) is any solution of (1). Note that r, s are integers and r>s>0. Treat r and s as the generators of the Pythagorean triangle $T(\alpha, \beta, \gamma)$, where $\alpha = 2rs$, $\beta = r^2 - s^2$, $\gamma = r^2 + s^2$. Let A and P represented its area and perimeter respectively. Then, this Pythagorean triangle T is such that

   $(a) 30\beta - \alpha - 29\gamma \equiv 0 (\mathrm{mod}\, 11^t)$

   $(b) \gamma - 31\alpha + \frac{120A}{P} \equiv 0 (\mathrm{mod}\, 11^t)$
2. Let x and y be taken as the sides of a rectangle R whose length of the diagonal, perimeter and area are denoted by L, P and A respectively. Note that

   $(i) 6[L^2 + 11^t]$ is a nasty number

   $(ii) P^2 - 8A = 4L^2$

CONCLUSION

In this paper, the integer solutions of the Pell equation $x^2 = 15y^2 - 11^t$ where t odd are obtained. For the case t even, we find that there is no integer solution as the negative Pell equation $x^2 = 15y^2 - 1$ has no integer solution. To conclude, one may search for integer solutions of other choices of negative Pell equations

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