**Research Article**

**Some Results Of Fixed Point Theorem In Dislocated Quasi-Metric Spaces Of Integral Type**

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**Abstract:** The purpose of this paper is to the study of fixed point theorems in dislocated quasi-metric spaces of integral type and obtain some new results in it. Also the paper contains generalized fixed point theorems of F. M. Zeyada et al., C.T. Aage & J.N. Salunke in dislocated quasi-metric space in integral type  

**Keywords:** Fixed point theorem, Continuous Mapping, Complete metric space

**INTRODUCTION**

Let $X$ be a nonempty set and let $d : X \times X \to [0, \infty)$ be a function satisfying the following conditions: 

(i) $\int \xi(t)dt = 0 \Rightarrow x = y$  

(ii) $\int \xi(t)dt \leq \int \xi(t)dt + \int \xi(t)dt = , \text{ for all } x, y, z \in X.$

Then $d$ is called a dislocated quasi-metric on $X$. If $d$ satisfies $\int \xi(t)dt = 0$, then it is called a quasi-metric on $X$. If $d$ satisfies $d(x, y) = d(y, x)$, then it is called a dislocated metric.

**Definition 1.1** Let $X$ be a nonempty set and $p : X \times X \to [0, \infty)$ be a function. We say $p$ is a partial metric on $X$ if it satisfies the following axioms: 

(i) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,  
(ii) $p(x, x) \leq p(x, y)$  
(iii) $p(x, y) = p(y, x)$  
(iv) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$  
for all $x, y, z \in X$.

Observe that any partial metric is a dislocated metric. Ultra metric $d$ on $X$ is a metric on $X$ with condition $\int \xi(t)dt \leq \int \xi(t)dt + \int \xi(t)dt$. The study of partial metric spaces and generalized ultra metric spaces have application in theoretical computer science\cite{2,3}. The notion of the dislocated topologies is useful in the context of logic programming. Recently, Zeyada et al.\cite{1} have established a fixed point theorem in a complete dislocated quasi-metric (dq-metric) space, as stated in the following lemma and theorem.

**Lemma 1.1** Let $(X, d)$ be a dq-metric space. If $f : X \to X$ is a contraction function, then $(f^n(x_0))$ is a cauchy sequence for each $x_0 \in X$. 

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**Theorem 1.1** Let \((X, d)\) be a complete dq-metric space and let \(f : X \rightarrow X\) be a continuous contraction function. Then \(f\) has a unique fixed point.

**Preliminaries**

**Definition 2.1** A sequence \(\{X_n\}\) in a dq-metric space (dislocated quasi-metric space) \((X, d)\) is called Cauchy if for given \(\varepsilon > 0\), \(\exists n_0 \in \mathbb{N}\) such that \(\forall m, n \geq n_0\), implies

\[
\int_{0}^{\infty} \xi(t) dt < \varepsilon \quad \text{or} \quad \int_{0}^{\infty} \xi(t) dt < \varepsilon
\]

\[
\min\{d(x, x_n), d(x, x_{n+1})\}
\]

i.e.

\[
\int_{0}^{\infty} \xi(t) dt < \varepsilon
\]

In the above definition if we replace \(\min\{d(x, x_n), d(x, x_{n+1})\}\)

\[
\max\{d(x, x_n), d(x, x_{n+1})\}
\]

By

\[
\int_{0}^{\infty} \xi(t) dt < \varepsilon,
\]

the sequence \(\{x_n\}\) is called “bi” Cauchy. Note that every bi Cauchy sequence is Cauchy.

**Definition 2.2** A sequence \(\{X_n\}\) dislocated quasi-converges to \(x\) if

\[
\lim_{n \rightarrow \infty} \int_{0}^{\infty} \xi(t) dt = \lim_{n \rightarrow \infty} \int_{0}^{\infty} \xi(t) dt = 0
\]

In this case \(x\) is called a dq-limit of \(\{x_n\}\).

**Proposition 2.1**. Every convergent sequence in a dq-metric space is bi ‘Cauchy.

**Proof.** Let \(\{x_n\}\) be a convergent sequence in a dq-metric space \((X, d)\) and \(x \in X\) be its dq-limit. That is,

\[
\lim_{n \rightarrow \infty} \int_{0}^{\infty} \xi(t) dt = \lim_{n \rightarrow \infty} \int_{0}^{\infty} \xi(t) dt = 0
\]

\[
d(x, x_n) \int_{0}^{\infty} \xi(t) dt < \varepsilon/2.\] Now \(n_0 = \max\{n_1; n_2\} \in \mathbb{N}\) is such that \(m, n \geq n_0\) implies \(d(x, x_n) \int_{0}^{\infty} \xi(t) dt \leq \varepsilon/2\).

Then \(\varepsilon > 0; \exists n_1; n_2 \in \mathbb{N}\) such that \(n \geq n_1\) and \(n \geq n_2\)

\[
d(x, x_n) \int_{0}^{\infty} \xi(t) dt + \int_{0}^{\infty} \xi(t) dt < \varepsilon/2 + \varepsilon/2 = \varepsilon\] and \(\int_{0}^{\infty} \xi(t) dt \leq \int_{0}^{\infty} \xi(t) dt \int_{0}^{\infty} \xi(t) dt < \varepsilon/2 + \varepsilon/2 = \varepsilon\)

Hence \(\{x_n\}\) is bi Cauchy.

Converse of proposition 2.1 may not be true. Proof of the following lemma is obvious

**Lemma 2.1**. Every subsequence of dq-convergent sequence to a point \(x_0\) is dq-convergent to \(x_0\).

**Definition 2.3** A dq-metric space \((X; d)\) is called complete if every Cauchy sequence in it is a dq-convergent.
Definition 2.4. Let \((X, d_1)\) and \((Y, d_2)\) be \(dq\)-metric spaces and let \(f : X \to Y\) be a function. Then \(f\) is continuous if for each sequence \(\{x_n\}\) which is \(d_1q\)-convergent to \(x_0\) in \(X\), the sequence \(\{f(x_n)\}\) is \(d_2q\)-convergent to \(f(x_0)\) in \(Y\).

MAIN RESULTS

Theorem 3.1 Let \((X, d)\) be a complete \(dq\)-metric space and suppose there exist non negative constants \(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\) with \(\alpha_1+\alpha_2+\alpha_3+2(\alpha_4+\alpha_5) < 1\). Let \(f : X \to X\) be a continuous mapping satisfying
\[
\int_0^1 \xi(t)dt \leq \alpha_1 \int_0^1 \xi(t)dt + \alpha_2 \int_0^1 \xi(t)dt + \alpha_3 \int_0^1 \xi(t)dt + \alpha_4 \int_0^1 \xi(t)dt + \alpha_5 \int_0^1 \xi(t)dt
\]
for all \(x, y \in X\). Then \(f\) has a unique fixed point.

**Proof:** Let \(\{x_n\}\) be a sequence in \(X\), defined as follows. Let \(x_0 \in X, f(x_0) = x_1, f(x_1) = x_2, \ldots, f(x_n) = x_{n+1}, \ldots\)

\[
\begin{align*}
\int_0^1 \xi(t)dt &\leq \alpha_1 \int_0^1 \xi(t)dt + \alpha_2 \int_0^1 \xi(t)dt + \alpha_3 \int_0^1 \xi(t)dt + \alpha_4 \int_0^1 \xi(t)dt + \alpha_5 \int_0^1 \xi(t)dt \\
\alpha_5 \int_0^1 \xi(t)dt &\leq (\alpha_1 + \alpha_2 + \alpha_4) \int_0^1 \xi(t)dt + (\alpha_3 + \alpha_4) \int_0^1 \xi(t)dt + \alpha_5 \int_0^1 \xi(t)dt \\
&= (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5) \int_0^1 \xi(t)dt + (\alpha_3 + \alpha_4 + \alpha_5) \int_0^1 \xi(t)dt
\end{align*}
\]

Therefore
\[
\int_0^1 \xi(t)dt \leq \lambda \int_0^1 \xi(t)dt
\]

where
\[
\lambda = \frac{\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5}{1 - \alpha_3 - \alpha_4 - \alpha_5}
\]

Similarly
\[
\begin{align*}
\int_0^1 \xi(t)dt &\leq \lambda \int_0^1 \xi(t)dt \\
\int_0^1 \xi(t)dt &\leq \lambda' \int_0^1 \xi(t)dt \\
\int_0^1 \xi(t)dt &\leq \lambda'' \int_0^1 \xi(t)dt
\end{align*}
\]

Since \(0 \leq \lambda < 1\), so for \(n \to \infty\), \(\lambda^n \to \infty\) we have \(d(x_n, x_{n+1}) \to 0\). Hence \(\{x_n\}\) is a Cauchy sequence in the
complete dislocated quasi-metric space \( X \), so there is a point \( t_0 \in X \), such that \( x_t \to t_0 \). Since \( f \) is continuous,

\[
f(t_0) = \lim f(x_t) = \lim x_{n+1} = t_0.
\]

Thus \( f(t_0) = t_0 \), so \( f \) has a fixed point.

**Uniqueness:** If \( x \in X \) is a fixed point of \( f \), then by (3.1)

\[
d(x, x) = \int_0^1 \xi(t) \, dt = \int_0^1 \xi(t) \, dt \\
\leq [\alpha_1 + \alpha_2 + \alpha_3 + 2(\alpha_4 + \alpha_5)] \int_0^1 \xi(t) \, dt
\]

which is true only if \( d(x, x) = 0 \), since \( 0 \leq \alpha_1 + \alpha_2 + \alpha_3 + 2(\alpha_4 + \alpha_5) < 0 \) and \( d(x, x) \geq 0 \). Thus \( d(x, x) = 0 \) for a fixed-point \( x \) of \( f \).

Let \( x, y \) be fixed point of \( f \). Then by (3.1)

\[
\int_0^1 \xi(t) \, dt = \int_0^1 \xi(t) \, dt \\
\leq \alpha_1 \int_0^1 \xi(t) \, dt + \alpha_2 \int_0^1 \xi(t) \, dt + \alpha_3 \int_0^1 \xi(t) \, dt + \alpha_4 \int_0^1 \xi(t) \, dt + \alpha_5 \int_0^1 \xi(t) \, dt
\]

\[
d(x, y) \leq (\alpha_1 + 2\alpha_5) \int_0^1 \xi(t) \, dt
\]

and from this it follows that \( d(x, y) = 0 \), since \( d(x, y) \geq 0 \).

Hence \( x = y \), i.e. uniqueness of the fixed point follows.

**Note:** If \( \alpha_2 = 0 = \alpha_3 \) in (3.1), then \( f \) becomes a contraction map and this shows that theorem 3.1 is a generalization of Theorem 1.1. Thus Theorem 3.1 is a generalization of Banach contraction principle.

**Theorem 3.2**

Let \( (X, d) \) be a complete dis quasi-metric space and let \( f^*: X \to X \) be a continuous mapping satisfying

\[
\int_0^1 \xi(t) \, dt \leq \alpha \int_0^1 \xi(t) \, dt + \beta \int_0^1 \xi(t) \, dt
\]

for all \( x, y \in X \). If \( 0 \leq \beta < 1 \) such that \( \alpha + 2\beta < 1 \) then \( f \) has a unique fixed point.
Proof: Let \( \{x_n\} \) be a sequence in \( X \), defined as follows. Let \( x_0 \in X, f(x_0) = x_1, f(x_1) = x_2, \ldots, f(x_n) = x_{n+1}, \ldots \)

\[
\begin{align*}
\int_0^{d(X_n, X_{n+1})} \xi(t) dt & \leq \int_0^{d(X_{n-1}, f(X_n))} \xi(t) dt \\
& \leq \alpha \int_0^{\max\{d(X_n, X_{n+1}), d(X_{n-1}, X_n)\}} \xi(t) dt + \beta \int_0^{\max\{d(X_{n-1}, X_{n+1}), d(X_{n-2}, X_{n-1})\}} \xi(t) dt \\
& \equiv \lambda \int_0^{\max\{d(X_n, X_{n+1}), d(X_{n-1}, X_n)\}} \xi(t) dt \\
& \equiv \beta \int_0^{\max\{d(X_{n-1}, X_{n+1}), d(X_{n-2}, X_{n-1})\}} \xi(t) dt \\
& \equiv \lambda \int_0^{\max\{d(X_n, X_{n+1}), d(X_{n-1}, X_n)\}} \xi(t) dt \\
& \equiv \beta \int_0^{\max\{d(X_{n-1}, X_{n+1}), d(X_{n-2}, X_{n-1})\}} \xi(t) dt
\end{align*}
\]

When

\[
\begin{align*}
\int_0^{d(X_n, X_{n+1})} \xi(t) dt &= \int_0^{d(X_{n-1}, X_n)} \xi(t) dt \\
& \leq \alpha \int_0^{d(X_{n-1}, X_n)} \xi(t) dt + \beta \int_0^{d(X_{n-1}, X_n)} \xi(t) dt \\
& = \lambda \int_0^{d(X_{n-1}, X_n)} \xi(t) dt \\
& \equiv \lambda \int_0^{d(X_{n-1}, X_n)} \xi(t) dt \\
& \equiv \beta \int_0^{d(X_{n-1}, X_n)} \xi(t) dt
\end{align*}
\]

Thus

\[
\begin{align*}
\int_0^{d(X_n, X_{n+1})} \xi(t) dt & \leq \lambda \int_0^{d(X_{n-1}, X_n)} \xi(t) dt \\
& \leq \beta \int_0^{d(X_{n-1}, X_n)} \xi(t) dt
\end{align*}
\]

Since \( 0 \leq \gamma < 1, \text{as} \ n \to \infty, \gamma^n \to \infty. \) Hence \( \{x_n\} \) is a \( d \)-cauchy sequence in \( X \). Thus \( \{x_n\} \) dislocated quasi-converges to some \( t_0. \) Since \( f \) is continuous, we have \( f(t_0) = \lim_{n \to \infty} x_n = \lim_{n+1} x_{n+1} = t_0 \)

Thus \( f(t_0) = t_0 \) that is \( f \) has a fixed point \( t_0. \)

Case-2

When

\[
\begin{align*}
\int_0^{d(X_n, X_{n+1})} \xi(t) dt &= \int_0^{d(X_{n-1}, X_n)} \xi(t) dt \\
& \leq \int_0^{d(X_{n-1}, X_n)} \xi(t) dt \\
& \leq \alpha \int_0^{d(X_{n-1}, X_n)} \xi(t) dt + \beta \int_0^{d(X_{n-1}, X_n)} \xi(t) dt \\
& = (1 - \beta) \int_0^{d(X_{n-1}, X_n)} \xi(t) dt + \beta \int_0^{d(X_{n-1}, X_n)} \xi(t) dt
\end{align*}
\]
\[
\int_0^{x(t)} dt \leq \left(\frac{\alpha + \beta}{1 - \beta}\right) \int_0^{y(t)} dt
\]

\[
\int_0^{x(t)} dt \leq \delta \int_0^{y(t)} dt \quad \text{where} \quad \delta = \frac{\alpha + \beta}{1 - \beta} < 1
\]

**Uniqueness:** Let \( x \) be a fixed point of \( f \), then by (3.2)

\[
\int_0^{\xi(t)} dt = \max \left\{ \int_0^{\xi(t)} dt \leq \lambda, \int_0^{y(t)} dt \right\}
\]

i.e.

\[
\int_0^{\xi(t)} dt \leq \lambda \int_0^{\xi(t)} dt
\]

which gives \( d(x; x) = 0 \), since \( 0 \leq \gamma < 1 \) and \( d(x; x) \geq 0 \). Thus

\[ d(x; x) = 0 \] if \( x \) is a fixed point of \( f \).

Let \( x, y \in X \) be fixed points of \( f \). That is \( fx = x; fy = y \). Then by (3.2),

\[
\int_0^{\xi(t)} dt = \max \left\{ \int_0^{\xi(t)} dt \leq \alpha \int_0^{\xi(t)} dt + \beta \int_0^{\xi(t)} dt \right\}
\]

\[ = (\alpha + 2\beta) \int_0^{\xi(t)} dt \]

which is true only if \( d(x; y) = 0 \) since \( d(x; x) = d(y; y) \geq 0 \leq \gamma < 1 \).

Similarly \( d(y; x) = 0 \) and hence \( x = y \).

Thus a fixed point of \( f \) is unique.

**Note:** If \( d \) is a partial metric on \( X \), then \( (X; d) \) becomes a dq-metric space. Hence we consider \( (X; d) \) in Theorem 3.1 and 3.2 as a partial metric space, then the conclusion follows.

**Theorem 3.3** Let \( (X; d) \) be a complete partial metric space and let \( f : X \rightarrow X \) be a continuous mapping satisfying

\[
\max \left\{ \int_0^{\xi(t)} dt \leq \alpha \int_0^{\xi(t)} dt + \beta \int_0^{\xi(t)} dt \right\}
\]

for all \( x, y \in X \). If \( 0 \leq \alpha, \beta < 1 \) such that \( \alpha + 2\beta < 1 \) then \( f \) has a unique fixed point.

It can be proved easily.

**REFERENCES**

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