Research Article

Integer Points on the Hyperbola \( x^2 - 6xy + y^2 + 4x = 0 \)

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Abstract: The binary quadratic equation \( x^2 - 6xy + y^2 + 4x = 0 \) representing hyperbola is considered. Different patterns of solutions are obtained. A few interesting recurrence relations satisfied by \( x \) and \( y \) are exhibited.

Keywords: binary quadratic, hyperbola, integer solutions

INTRODUCTION:

The binary quadratic equation offers an unlimited field for research because of their variety [1-5]. In this context one may also refer [6-19]. This communication concerns with yet another interesting binary quadratic equation \( x^2 - 6xy + y^2 + 4x = 0 \) for determining its infinitely many non-zero integral solutions. Also a few interesting relations are presented.

METHOD OF ANALYSIS

The hyperbola under consideration is

\( x^2 - 6xy + y^2 + 4x = 0 \)  \hspace{1cm} (1)

Different patterns of solutions for (1) are illustrated below:

Pattern: 1

Introducing the linear transformations \((X \neq T \neq 0)\),

\[ x = X + T \quad \text{and} \quad y = X - T \]  \hspace{1cm} (2)

In (1), it becomes

\[ Y^2 = 2Z^2 - 1 \]  \hspace{1cm} (3)

Where, \( Y = 4T + 1 \) and \( Z = 2X - 1 \) \hspace{1cm} (4)

The smallest positive integer solution of (3) is \( Z_0 = 1 \) and \( Y_0 = 1 \)

To find the other solution of (3), consider the pellian equation

\[ Y^2 = 2Z^2 + 1 \]

whose general solution \((Y_n, Z_n)\) is given by

\[ Y_n = \frac{1}{2} \left[ (3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1} \right] \]

\[ Z_n = \frac{1}{2\sqrt{2}} \left[ (3 + 2\sqrt{2})^{n+1} - (3 - 2\sqrt{2})^{n+1} \right] \]

Applying Brahmagupta Lemma between \((Y_0, Z_0)\) and \((Y_n, Z_n)\), the general solutions to (3) are given by,

\[ Y_{n+1} = Y_0Y_n + 2Z_0Z_n \]
\[ Z_{n+1} = Z_0Y_n + Y_0Z_n \]

In view of (4), we have
\[ X_{n+1} = \frac{1}{2} (Y_n + Z_n + 1) \]
\[ T_{n+1} = \frac{1}{4} (Y_n + 2Z_n - 1) \]

Employing (2), the values of \( x \) and \( y \) satisfying (1) are given by

\[ x_{n+1} = \frac{1}{8} \left( (3 + 2\sqrt{2})^{y_{n+1}} + (3 - 2\sqrt{2})^{y_{n+1}} \right) + \frac{1}{4}, \quad n = 1, 3, 5, \ldots \]
\[ y_{n+1} = \frac{1}{8} \left( (3 + 2\sqrt{2})^{y_{n+1}} + (3 - 2\sqrt{2})^{y_{n+1}} \right) + \frac{3}{4}, \quad n = 1, 3, 5, \ldots \]

Properties

- \( 4x_{n+1} - 140x_{n+2} + 24x_{n+1} = -28 \)
- \( 6x_{n+2} - x_{n+1} - x_{n+3} = 1 \)
- \( 34x_{n+3} - x_{n+5} - x_{n+1} = 8 \)
- \( 6x_{n+4} - x_{n+3} - x_{n+5} = 1 \)
- \( y_{n+5} - 34y_{n+3} + y_{n+1} = -24 \)
- \( 70y_{n+2} - 2y_{n+4} - 12y_{n+1} = 48 \)
- \( y_{n+4} + y_{n+2} - 6y_{n+3} = -3 \)
- \( y_{n+5} - 6y_{n+4} + y_{n+3} = 3 \)
- Each of the expressions represents a Nasty Number:
  - \( 48x_{2n} + 18 \)
  - \( 48y_{2n+2} - 24 \)
- Each of the expressions represents a cubic integer:
  - \( 8x_{3n+5} + 24x_{n+1} - 8 \)
  - \( 8y_{3n+3} + 24y_{n+1} - 24 \)
- Each of the expressions represents a bi-quadratic integer:
  - \( 8x_{4n+7} + 256x_{n+1}^2 - 128x_{n+1} + 12 \)
  - \( 8y_{4n+4} + 256y_{n+1}^2 - 384y_{n+1} - 136 \)

Note

Instead of (2), if we consider the linear transformations \((X \neq T \neq 0)\),
\[ x = X - T \quad \text{and} \quad y = X + T \]

Then, the corresponding integer solutions to (1) are obtained as

\[ x_{n+1} = \frac{1}{8} \left( (3 + 2\sqrt{2})^{y_{n+1}} + (3 - 2\sqrt{2})^{y_{n+1}} \right) + \frac{3}{4}, \quad n = 0, 2, 4, \ldots \]
\[ y_{n+1} = \frac{1}{8} \left( (3 + 2\sqrt{2})^{y_{n+1}} + (3 - 2\sqrt{2})^{y_{n+1}} \right) + \frac{1}{4}, \quad n = 0, 2, 4, \ldots \]

The recurrence relations satisfied by \( x \) and \( y \) are given by

\[ x_{n+1} + 3 = 6x_{n+2} - x_{n+3}; \quad x_1 = 5, x_3 = 145 \]
\[ y_{n+1} + 1 = 6y_{n+2} - y_{n+3}; \quad y_1 = 1, y_3 = 25 \]

Some numerical examples of \( x \) and \( y \) satisfying (1) is given in the following table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_{n+1} )</th>
<th>( y_{n+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>145</td>
<td>25</td>
</tr>
<tr>
<td>4</td>
<td>4901</td>
<td>841</td>
</tr>
<tr>
<td>6</td>
<td>166465</td>
<td>28561</td>
</tr>
</tbody>
</table>
From the above table relations observed are as follows:

- $x_{n+1}$ and $y_{n+1}$ are always odd
- $y_{6n-5}$ and $y_{6n-1}$ are perfect squares
- $6y_{6n-1}$ is a Nasty number
- $x_{6n-5} \equiv 0 \pmod{5}$
- $y_{6n-3} \equiv 0 \pmod{5}$
- $x_{6n-3} \equiv 0 \pmod{5}$

**Pattern: 2**

Treating (1) as a quadratic in $x$ and solving for $x$, we get

$$x = 3y^2 - 2 \pm 2\sqrt{2y^2 - 3y + 1}$$  \hspace{1cm} (5)

Let $\alpha^2 = 2y^2 - 3y + 1$  \hspace{1cm} (6)

Substituting $y = \frac{Y + 3}{4}$  \hspace{1cm} (7)

In (6), we have

$$Y^2 = 8\alpha^2 + 1$$

whose general solution is given by,

$$Y_n = \frac{1}{2} \left[ (3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1} \right]$$  \hspace{1cm} (8)

$$\alpha_n = \frac{1}{4\sqrt{2}} \left[ (3 + 2\sqrt{2})^{n+1} - (3 - 2\sqrt{2})^{n+1} \right]$$  \hspace{1cm} (9)

From (7) and (8), we have

$$y_n = \frac{1}{8} \left[ (3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1} \right] + \frac{3}{4}$$  \hspace{1cm} (10)

Substituting (9) and (10) in (5) and taking the positive sign, the corresponding integer solutions to (1) are given by

$$x_n = \frac{1}{8} \left[ (3 + 2\sqrt{2})^{n+2} + (3 - 2\sqrt{2})^{n+2} \right] + \frac{1}{4}, \quad n = 1, 3, 5, \ldots$$

$$y_n = \frac{1}{8} \left[ (3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1} \right] + \frac{3}{4}, \quad n = 1, 3, 5, \ldots$$

**Properties**

- $48x_{2n+2}$ is a Nasty Number
- $8x_{3n+4} + 24x_n - 8$ is a Cubical integer
- $8x_{4n+6} + 256x_n^2 - 128x_n + 12$ is a Bi-quadratic integer
- Define $\beta = 4y_n^2 - 3$ and $\gamma = x_n - 3y_n^2 + 2$. Note that the pair $(\beta, \gamma)$ satisfies the hyperbola $\beta^2 = 2\gamma^2 + 1$
- $2x_{2n} = (4y_n^2 - 3)^2$

Also, taking the negative sign in (5), the other set of solutions to (1) is given by

$$x_n = \frac{1}{8} \left[ (3 + 2\sqrt{2})^{n+2} + (3 - 2\sqrt{2})^{n+2} \right] + \frac{1}{4}, \quad n = 1, 3, 5, \ldots$$
\[
y_n = \frac{1}{8} \left[ \left(3 + 2\sqrt{2}\right)^{n+1} + \left(3 - 2\sqrt{2}\right)^{n+1} \right] + \frac{3}{4}, \quad n = 1, 3, 5, \ldots
\]

In addition, the above two sets of solutions satisfy the following properties:

- \[6y_{n+2} - y_{n+3} - y_{n+1} = 3\]
- \[140y_{n+4} - 4y_{n+3} - 24y_n = 84\]
- \[y_{n+2} + y_n - 6y_{n+1} = -3\]
- \[34y_{n+2} - y_{n+4} - y_n = 24\]
- \[x_{n+4} + x_{n+2} - 6x_{n+3} = -1\]
- \[70x_{n+1} - 2x_{n+3} - 12x_n = 14\]
- \[34x_{n+2} - x_{n+4} - x_n = 8\]
- \[x_n + x_{n+2} - 6x_{n+1} = -1\]
- \[48y_{2n+1} - 24\] is a Nasty Number
- \[8y_{3n+2} + 24y_n - 24\] is a Cubical integer
- \[8y_{4n+3} + 256y_n^2 - 384y_n - 136\] is a Bi-quadratic integer

**Pattern: 3**

Treating (1) as a quadratic in \(y\) and solving for \(y\), we get

\[
y = 3x \pm 2\sqrt{2}x^2 - x
\]

Let \[\alpha^2 = 2x^2 - x\] \hspace{1cm} (11)

Let \[\alpha^2 = 2x^2 - x\] \hspace{1cm} (12)

Substituting \(x = \frac{X + 1}{4}\) \hspace{1cm} (13)

In (12), we have

\[X^2 = 8\alpha^2 + 1\]

whose general solution is given by,

\[X_n = \frac{1}{2} \left[ \left(3 + 2\sqrt{2}\right)^{n+1} + \left(3 - 2\sqrt{2}\right)^{n+1} \right] \]

\[\alpha_n = \frac{1}{4\sqrt{2}} \left[ \left(3 + 2\sqrt{2}\right)^{n+1} - \left(3 - 2\sqrt{2}\right)^{n+1} \right] \]

From (13) and (14), we have

\[x_n = \frac{1}{8} \left[ \left(3 + 2\sqrt{2}\right)^{n+1} + \left(3 - 2\sqrt{2}\right)^{n+1} \right] + \frac{1}{4} \]

(16)

Substituting (15) and (16) in (11) and taking the positive sign, the corresponding integer solutions to (1) are given by

\[x_n = \frac{1}{8} \left[ \left(3 + 2\sqrt{2}\right)^{n+2} + \left(3 - 2\sqrt{2}\right)^{n+2} \right] + \frac{1}{4}, \quad n = 0, 2, 4, \ldots\]

\[y_n = \frac{1}{8} \left[ \left(3 + 2\sqrt{2}\right)^{n+2} + \left(3 - 2\sqrt{2}\right)^{n+2} \right] + \frac{3}{4}, \quad n = 0, 2, 4, \ldots\]

**Properties**

- \[48y_{2n+2} - 24\] is a Nasty Number
- \[8y_{3n+4} + 24y_n - 24\] is a Cubical integer
- \[8y_{4n+6} + 256y_n^2 - 384y_n + 136\] is a Bi-quadratic integer

\[2y_{2n} - 1 = (4x_n - 1)^2\]

Also, taking the negative sign in (11), the other set of solutions to (1) is given by
\[
x_n = \frac{1}{8}\left[ (3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n \right] + \frac{1}{4}, \quad n = 0, 2, 4, \ldots
\]
\[
y_n = \frac{1}{8}\left[ (3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n \right] + \frac{3}{4}, \quad n = 0, 2, 4, \ldots
\]

In addition, the above two sets of solutions satisfy the following properties:

- \(48x_{2n+1}\) is a Nasty Number
- \(8x_{3n+2} + 24x_n - 8\) is a Cubical integer
- \(8x_{4n+3} + 256x_n^2 - 128x_n + 12\) is a Bi-quadratic integer

**CONCLUSION**

As the binary quadratic equations are rich in variety, one may consider other choices of hyperbolas and search for their non-trivial distinct integral solutions along with the corresponding properties.

**REFERENCES**

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