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Existence of Periodic Solutions of Nonlinear Strain Waves Equations

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Abstract

In this paper, we consider the existence of time periodic solutions of the nonlinear strain wave's equation. The method we use is similar to the existence of periodic solution of the navier-stokes equations. Firstly, by Leray-Schauder fixed point theorem, we show the existence of approximate solutions of the modified nonlinear strain wave's equation, then we show the convergence of the approximate solutions, and we also get the uniqueness of the solution to the modified equation.
Keywords: nonlinear strain waves equation, time periodic solutions, existence, and uniqueness.

INTRODUCTION

Refer literature [1], we know the Nonlinear strain waves equation was introduced by Xianyun Du in 2011. Before that, in [2] and [3], we have learned the concept of random attractors. In 1997, Hisako Kato wrote about the existence of periodic solutions of the Navier-Stokes equations in [4]. Therefore, we have combined many articles and combined with the article by Guo B and Xianyun Du in [5], we want to study about the existence of time periodic solutions of the nonlinear strain wave's equation. We motivated by the ideas in [1, 4, 6, 7], we can accomplish this paper. As we all known, in some problems of nonlinear wave propagation in waveguides, the interaction of waveguides, the external medium and the possibility of energy exchange through lateral surface of waveguide cannot be neglected. When the energy exchange between the rod and the medium is considered, there is a dissipation of deformation wave in the viscous external medium. The general cubic double dispersion equation can be derived from Hamilton principle:

$$w_{tt} - w_{xx} = \frac{1}{4}(aw^3 + 6w^2 + bw_{tt} - cw_{xx} + dw_t)_{xx} \#(1.1)$$

where a, b, c, d are some positive constants depending on Young modulus E_0.

In this paper, we concerned the existence and uniqueness of periodic of nonlinear strain waves equations:

$$w_{tt} - (\alpha w_{tt} + \beta w_t)_{xx} = \left(w - \gamma w_{xx} + f(w)\right)_{xx} + g(x,t) \#(1.2)$$

$$w(x,t) = \mathrm{w}(x, t+T) \#(1.3)$$

where α, β, γ are positive constants, f is a sufficiently smooth real valued function with $f(0) = 0$, let the given external force $g(x,t)$ be periodic in t with the same period T.

To describe our theorems accurately, we introduce some function spaces and notion. We define

$$C_{0,\sigma}^{\infty} \equiv \{\epsilon \in C_0^{\infty}(\Omega)\}$$

where Ω is a bounded domain in the N-dimentional Euclidean space R^N with smooth boundary $\partial\Omega$. In addition, we define H_σ as the closure of $C_{0,\sigma}^{\infty}$ in $L_2(\Omega)$, and $H_{0,\sigma}^1$ as the closure of $C_{0,\sigma}^{\infty}$ in $H^1(\Omega)$. Throughout this paper, $L_2(\Omega)$ represents the Hilbert space equipped with the inner product

$$(u,v) = \sum_{i=1}^{N} \int u^i v^i dx$$

We denote the $L_2(\Omega)$-norm by $\|.\|$. $L_2(\Omega)$ and H^m are usual Sobolev spaces. Then let X be a Banach space. We denote by $C^k(T;X)$ the set of X-valued T-periodic functions on R^1 with continuous derivatives up to order k. Next let us define the norm

$$\|f\|_{C^k(T;X)} = \sup_{0 \le t \le T} \left\{ \sum_{i=0}^{k} \left\| D_t^i f(t) \right\|_X \right\}$$

We denote by $L_P(T;X)$ $(1 \le P \le \infty)$ the set of T-periodic X-valued measurable functions f on R^1 such that

$$\|f\|_{L_P(T;X)} = \left(\int_0^T \|f(t)\|_X^P \right)^{1/P} < +\infty \quad (1 \le P \le \infty),$$
$$\|f\|_{L_\infty(T;X)} = \sup_{0 \le t \le T} \|f(t)\|_X < +\infty$$

We denote by $W^{k,p}(T;X)$ the set of functions f which belong to $L_P(T;X)$ together with their derivatives up to order k, and in particular we write $H^k(T;X) = W^K(T;X)$ when X is a Hilbert space, and let A be the unbounded linear operator defined by $\mathrm{A}w = -D_x^2 w$.

It is convenient to introduce a transformation:

$$\mathrm{u} = w_t + \varepsilon \mathrm{w} \Rightarrow u_t = w_{tt} + \varepsilon w_t = w_{tt} + \varepsilon(u - \varepsilon w)$$

so, we have:

$$w_t = \mathrm{u} - \varepsilon \mathrm{w}$$
$$w_{tt} = u_t - \varepsilon u + \varepsilon^2 w$$

where ε is a positive. The problem (1.2) → (1.3) is equivalent to:

$$u_t - \varepsilon \mathrm{u} + \varepsilon^2 w - \alpha A(u_t - \varepsilon u + \varepsilon^2 w) - \beta A(u - \varepsilon w) = Aw - \gamma A^2 w + Af(w) + g(x,t)$$

the calculation is:

$$u_t - \varepsilon u + \varepsilon^2 w - \alpha A u_t + \alpha\varepsilon A u - \alpha\varepsilon^2 A w - \beta A u + \beta\varepsilon A w = Aw - \gamma A^2 w + Af(w) + g(x,t)$$

and then we can get:

$$u_t + (\alpha\varepsilon - \beta)Au - \alpha A u_t - \varepsilon u - (1 + \alpha\varepsilon^2 - \beta\varepsilon)Aw + \varepsilon^2 w + \gamma A^2 w - Af(w) = g(x,t)\#(1.4)$$
$$(w,u)(x,t) = (w,u)(x,t+T)\#(1.5)$$

Approximate solutions

In this section, we are going to show the existence of approximate solutions of (1.4) → (1.5). Let $\omega_i (i = 1,2,\dots)$ be the completely orthonormal system in H_σ consisting of the eigenfunctions of A with a homogeneous Dirichlet boundary condition on Ω. We consider as follows:

$$(u_{nt} + (\alpha\varepsilon - \beta)Au_n - \alpha A u_{nt} - \varepsilon u_n - (1 + \alpha\varepsilon^2 - \beta\varepsilon)Aw_n + \varepsilon^2 w_n + \gamma A^2 w_n - Af(w_n), \omega_i)$$
$$= (g(x,t), \omega_i), (i = 1,2,\dots,n)\#(2.1)$$
$$(w_n, u_n)(x,t) = (w_n, u_n)(x, t+T)\#(2.2)$$

where $w_n(t) = \sum_{i=1}^n a_{in}(t)\omega_i$, $u_n(t) = \sum_{i=1}^n b_{in}(t)\omega_i$.

Let W_n be the subspace of H_σ spanned by $\omega_1, \omega_2, \dots, \omega_n$.It is well known that for any $v_n(t) = \sum_{i=1}^n c_{in}(t)\omega_i \in C^1(T;W_n)$, there exists a unique T-periodic solution of the linear equation:

$$(u_{nt} + (\alpha\varepsilon - \beta)Au_n - \alpha A u_{nt} - \varepsilon u_n - (1 + \alpha\varepsilon^2 - \beta\varepsilon)Aw_n + \varepsilon^2 w_n + \gamma A^2 w_n, \omega_{ni})$$
$$= (Af(v_n) + g(x,t), \omega_i)\#(2.3)$$

With the mapping $\mathrm{F}: (u_n, v_n) \to (u_n, w_n)$ continuous and compact in $C^1(T, W_n)$, we shall prove the existence of the solution of (2.1) → (2.2), and by applying the Leray-Schauder fixed point theorem, it is only required to show the boundedness:

$$\sup_t \left(\|w_n(t)\|_{H_2} + \|u_n(t)\|_{L_2} \right) \le c$$

for all possible solutions of(2.1) → (2.2) replaced by $\delta(Af(w_n))(0 \le \delta \le 1)$ instead of nonlinear terms $Af(w_n)$ respectively, where c is a constant independent of δ.

Lemma 2.1. Suppose f is a nonlinear function satisfying conditions: for all m ∈ R, we have

$$f(m)\mathrm{m} \ge c_1 F(m) \ge c_2 |m|^{2p+2} \ge 0\#(2.4)$$

$$|f(m)| \le c_3(|m|^{2p+1} + |m|)\#(2.5)$$

where $\mathrm{F}(m) = \int_0^m f(s)ds$ and $c_i (i = 1,2,3)$ are positive constants.

and for $g \in H^{-1}(0.l)$,we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u_n\|_{-1,2}^2+\alpha\|u_n\|^2+\varepsilon^2\|w_n\|_{-1,2}^2+(1+\alpha\varepsilon^2-\beta\varepsilon)\|w_n\|^2+\gamma\|\nabla w_n\|^2+2\int_0^l F(w_n)dx\right)$$
$$+\lambda\left(\|u_n\|^2+\alpha\|u_n\|^2+\varepsilon^2\|w_n\|^2+(1+\alpha\varepsilon^2-\beta\varepsilon)\|w_n\|^2+\gamma\|\nabla w_n\|^2+2\int_0^l F(w_n)dx\right)$$
$$\le c_4 \#(2.6)$$

Proof: Taking the inner product of (1.4) with $(-\Delta)^{-1}u$ and using $u = w_t + \varepsilon w$, we have

$$\frac{d}{dt}\zeta(t)+\eta(t)=0\#(2.7)$$

where

$$\zeta(t)=\frac{1}{2}(\|u_n\|_{-1,2}^2+\alpha\|u_n\|^2+\varepsilon^2\|w_n\|_{-1,2}^2+(1+\alpha\varepsilon^2-\beta\varepsilon)\|w_n\|^2+\gamma\|\nabla w_n\|^2$$
$$+2\int_0^l F(w_n)dx)\#(2.8)$$

$$\eta(t)=-\varepsilon\|u_n\|_{-1,2}^2+(\beta-\alpha\varepsilon)\|u_n\|^2+\varepsilon^2\|w_n\|_{-1,2}^2+(1+\alpha\varepsilon^2-\beta\varepsilon)\varepsilon\|w_n\|^2+\gamma\varepsilon\|\nabla w_n\|^2$$
$$+\varepsilon\int_0^l f(w_n)w_n dx\,\#(2.9)$$

choose δ and ε such that

$$0<\delta\le min\left\{\frac{c_1}{2},1\right\},0<\varepsilon\le min\left\{\frac{1}{2\beta},\frac{\beta\lambda_1}{(1+\delta)(1+\alpha\lambda_1)}\right\}\#(2.10)$$

ere c_1 is defined in (2.4) and λ_1 is the first eigenvalue of A. It follows from (2.4) and (2.5) that

$$\varepsilon\int_0^l f(w_n)w_n dx-\delta\varepsilon\int_0^l F(w_n)\,dx\ge\varepsilon(c_1-\delta)c_2\|w_n\|_{2p+2}^{2p+2}$$

using (2.10) and computing, we get

$$\eta(t)-\delta\varepsilon\zeta(t)$$
$$\ge\left(\beta\lambda_1-\varepsilon(1+\delta)(1+\alpha\lambda_1)\right)\|u_n\|_{-1,2}^2+\varepsilon^3(1-\delta)\|w_n\|_{-1,2}^2+\varepsilon(1+\alpha\varepsilon^2-\beta\varepsilon)(1-\delta)\|w_n\|^2$$
$$+\gamma\varepsilon(1-\delta)\|\nabla w_n\|^2$$
$$+\varepsilon c_2(c_1-2\delta)\|w_n\|_{2p+1}^{2p+1}-c\|g\|_{-1,2}^2\ge-c\|g\|_{-1,2}^2\#(2.11)$$

and

$$a\left(\|Y(t)\|_{E_0}^2+\|w_n(t)\|_{2p+2}^{2p+2}\right)\le\zeta(t)\le b\left(\|Y(t)\|_{E_0}^2+\|w_n(t)\|_{2p+2}^{2p+2}\right)\#(2.12)$$

where

$$a=\frac{1}{2}min\{\alpha,(1+\alpha\varepsilon^2-\beta\varepsilon),\gamma,2c_2/c_1\}$$

and

$$b=\frac{1}{2}max\{\alpha+1/\lambda_1,\varepsilon^2/\lambda_1,(1+\alpha\varepsilon^2-\beta\varepsilon),2c_3/c_1,\gamma\}$$

by (2.7) and (2.11) , we have

$$\frac{d}{dt}\zeta(t)+\lambda\zeta(t)\le c_4\#(2.13)$$

where $\lambda=\delta\varepsilon>0$.
from the definition of $\zeta(t)$, we know that $\zeta(t)$ is a T-periodic function.
next, integrating (2.13) from 0 to T, we can get

$$\lambda\int_0^T\zeta(t)\le\int_0^T c_4=Tc_4\#(2.14)$$

and there exists $t^*\in[0,T]$, such that

$$\lambda\zeta(t^*)\le c_4\#(2.15)$$

integrating (2.13) again from t^* to $t+T(t\in[0,T])$, we can see

$$\int_{t^*}^{t+T}\frac{d}{dt}\zeta(t)\,dt+\int_{t^*}^{t+T}\lambda\zeta(t)dt\le\int_{t^*}^{t+T}c_4dt$$
$$\Rightarrow\zeta(t+T)-\zeta(t^*)+\int_{t^*}^{t+T}\lambda\zeta(t)dt\le c_4(t+T-t^*)$$
$$\Rightarrow\zeta(t)-\zeta(t^*)\le 2Tc_4$$

from (2.15) , we can obtain

$$\zeta(t) \le 2Tc_4 + \lambda^{-1}c_4 \#(2.16)$$

from (2.4), (2.8) and (2.16), we obtain

$$\sup_{0\le t\le T}(\|u_n\|^2 + \alpha\|u_n\|^2 + \varepsilon^2\|w_n\|^2 + (1+\alpha\varepsilon^2 - \beta\varepsilon)\|w_n\|^2 + \gamma\|\nabla w_n\|^2)$$
$$\le 4Tc_4 + 2\lambda^{-1}c_4 \#(2.17)$$

where C_4 is independent of N.
Set $4TC_4 + 2\lambda^{-1}C_4 = M_0$
from above all, we can easily get

$$\sup_t \|Y(t)\|_{E_0}^2 \le C(M_0)$$

and

$$\sup_t \|w_n(t)\|_{2p+2}^{2p+2} \le C(M_0)$$

so, we can see:

$$\sup_t \|w_n(t)\|_{H_2} \le C(M_0)$$

and then there has

$$\sup_t \|u_n(t)\|_{L_2} \le c(M_0)$$

The proof is complete.

Estimates of derivatives of higher order

In this section, we continue to estimate the high order derivatives of the solutions of $(2.1) \to (2.2)$. Before starting to prove our lemma on the uniform boundedness of $\|A^m w_n(t)\| (m = N/4 - 1/2)$, we note that we can choose the basis $\{\omega_i; i = 1,2,\ldots\}$ such that eigenfunctions ω_i of A are also eigenfunctions of A^m and that we can write

$$A\omega_i = \mu_i\omega_i, A^m\omega_i = \mu_i^m\omega_i \#(3.1)$$

where μ_i is the eigenvalue of A.

Lemma 3.1. let $w_n(t)$ be the solution of $(2.1) \to (2.2)$ give above. Suppose that:

$$\sup_{0\le t<+\infty} \|Ag(x,t)\|^2 = 2M \#(3.2)$$

Then we have

$$\|A^m w_n(t)\| \le \left(2T + \frac{1}{2}\varepsilon^{-1}T\right)(c(M_0) + M)\#(3.3)$$

Proof: considering $(2.1) \to (2.2)$ and(3.1), taking the inner product of (2.2) with$(-A)^{-1}A^{2m}u_n$, we can see

$$\frac{1}{2}\frac{d}{dt}\|A^m u_n\|^2 + (\beta - \alpha\varepsilon)\|A^m u_n\|^2 + \frac{1}{2}\frac{d}{dt}\alpha\|A^m u_n\|^2 - \varepsilon\|u_n\|^2 + \frac{1}{2}\frac{d}{dt}(1+\alpha\varepsilon^2-\beta\varepsilon)\|A^m w_n\|^2$$
$$+ (1+\alpha\varepsilon^2-\beta\varepsilon)\varepsilon\|A^m w_n\|^2 + \frac{1}{2}\frac{d}{dt}\varepsilon^2\|w_n\|^2 + \varepsilon^3\|w_n\|^2 + \frac{1}{2}\frac{d}{dt}\gamma\|A^{m+1}w_n\|^2 + \gamma\varepsilon\|A^{m+1}w_n\|^2$$
$$+ \int f(w_n)w_{nt}dx + \varepsilon\int f(w_n)w_n dx - (g(x,t), A^{2m}u_n)$$

and then we set

$$\zeta_1(t) = \frac{1}{2}(\|A^m u_n\|^2 + \alpha\|A^m u_n\|^2 + (1+\alpha\varepsilon^2-\beta\varepsilon)\|A^m w_n\|^2 + \varepsilon^2\|A^m w_n\|^2 + \gamma\|A^{m+1}w_n\|^2$$
$$+2\int F(w_n)dx \#(3.4)$$

$$\eta_1(t) = (\beta-\alpha\varepsilon)\|A^m u_n\|^2 - \varepsilon\|u_n\|^2 + (1+\alpha\varepsilon^2-\beta\varepsilon)\varepsilon\|A^m w_n\|^2 + \varepsilon^3\|A^m w_n\|^2 + \gamma\varepsilon\|A^{m+1}w_n\|^2$$
$$+\varepsilon\int f(w_n)w_n dx - (g(x,t), A^{2m}u_n)\#(3.5)$$
$$-(g(x,t), A^{2m}u_n) \ge -(\|A^m g(x,t)\|\|A^m u_n\|)$$

where we used Holder inequality and Young inequality.
Obviously, we have

$$\frac{d}{dt}\zeta_1(t) + \eta_1(t) = 0\#(3.6)$$

let $(3.5) - 2\,\varepsilon(3.4)$, we can see

$$\eta_1(t) - 2\varepsilon\zeta_1(t) \ge (\beta-\alpha\varepsilon)\|A^m u_n\|^2 - \varepsilon\|u_n\|^2 + \varepsilon\int f(w_n)w_n dx - (\|A^m g(x,t)\|\|A^m u_n\|) - \varepsilon\|A^m u_n\|^2$$
$$- \alpha\varepsilon\|A^m u_n\|^2 - 2\varepsilon\int F(w_n)dx$$
$$\ge (\beta - 2\alpha\varepsilon - \varepsilon)\|A^m u_n\|^2 - \varepsilon\|u_n\|^2 + \varepsilon\left(\int f(w_n)w_n dx - 2\int F(w_n)dx\right)$$

$$-\frac{1}{2}\|A^m g(x,t)\|^2 - \frac{1}{2}\|A^m u_n\|^2 \#(3.7)$$

where by using the spectral representation $\mathrm{A} = \int_\mu^\infty \lambda dE_\lambda\,, \mu > 0$, we obtain the inequality

$$\|A^a u_n\| \leq \mu^{a-b}\|A^b u_n\| \qquad (0 \leq \alpha \leq \beta)$$

choose $\mathrm{a} = \mathrm{m}, \mathrm{b} = 0$, so we have

$$\mu^m\|A^m u_n\| \geq \|u_n\|$$

then we can see

$$\|A^m u_n\|^2 \geq \mu^{-2m}\|u_n\|^2$$

set $\mu^{-2m}\beta = \varepsilon$,
by (2.4) (3.2) and (3.7), we can get

$$\eta_1(t) - 2\varepsilon\zeta_1(t) \geq -\mu^{-2m}\left(2\alpha\varepsilon + \varepsilon + \frac{1}{2}\right)\|u_n\|^2 - M \geq -c(M_0) - M\#(3.8)$$

by (3.6) → (3.8), we can have the conclusion:

$$\frac{d}{dt}\zeta_1(t) + 2\varepsilon\zeta_1(t) \leq \mathrm{c}(M_0) + \mathrm{M}\#(3.9)$$

from the definition of $\zeta_1(t)$, we know that $\zeta_1(t)$ is a T-periodic function.
so next integrating (3.9) from 0 to T, we can get

$$\int_0^T 2\varepsilon\zeta_1(t)dx \leq \int_0^T (c(M_0) + M)\mathrm{dx} = T(c(M_0) + M)$$

and there exists $t^* \in [0,T]$, such that

$$2\varepsilon\zeta_1(t^*) \leq T(c(M_0) + M) \Rightarrow \zeta_1(t^*) \leq \frac{1}{2}\varepsilon^{-1}T(c(M_0) + M)\#(3.10)$$

integrating (3.9) again from t^* to $\mathrm{t+T}(t \in [0,T])$, we can see

$$\int_{t^*}^{\mathrm{t+T}} \frac{d}{dt}\zeta_1(t)dx + \int_{t^*}^{\mathrm{t+T}} 2\varepsilon\zeta_1(t)dx \leq \int_{t^*}^{\mathrm{t+T}} (c(M_0) + M)\,\mathrm{dx}$$

$$\Rightarrow \zeta_1(t+T) - \zeta_1(t^*) + \int_{t^*}^{\mathrm{t+T}} 2\varepsilon\zeta_1(t)dx \leq (c(M_0) + M)(t + T - t^*)$$

$$\Rightarrow \zeta_1(t) - \zeta_1(t^*) \leq (c(M_0) + M)(t + T - t^*)$$

$$\Rightarrow \zeta_1(t) \leq 2T(c(M_0) + M) + \zeta_1(t^*) \leq 2T(c(M_0) + M) + \frac{1}{2}\varepsilon^{-1}T(c(M_0) + M)$$

so, we can see

$$\zeta_1(t) \leq \left(2T + \frac{1}{2}\varepsilon^{-1}T\right)(c(M_0) + M)\#(3.11)$$

from (3.4) and (3.11), we obtain:

$$\sup_t (\|A^m u_n\|^2 + \alpha\|A^m u_n\|^2 + (1 - \beta\varepsilon + \alpha\varepsilon^2)\|A^m w_n\|^2 + \varepsilon^2\|A^m w_n\|^2 + \gamma\|A^{m+1} w_n\|^2)$$

$$\leq \left(2T + \frac{1}{2}\varepsilon^{-1}T\right)(c(M_0) + M)\#(3.12)$$

set $\left(2T + \frac{1}{2}\varepsilon^{-1}T\right)\ (c(M_0) + M) = M_1$
from above all, we can easily get

$$\sup_t \|A^m w_n(t)\| \leq (2\varepsilon + 1)TM \Rightarrow \|A^m w_n(t)\| \leq \left(2T + \frac{1}{2}\varepsilon^{-1}T\right)\ (c(M_0) + M) \leq C(M_1)\#(3.13)$$

and

$$\sup_t \|A^m u_n(t)\| \leq (2\varepsilon + 1)TM \Rightarrow \|A^m u_n(t)\| \leq \left(2T + \frac{1}{2}\varepsilon^{-1}T\right)\ (c(M_0) + M) \leq C(M_1)\#(3.14)$$

Consequently, the proof of Lemma 3.1 is complete.

T-Periodic solutions

This part, we want to show the convergence of the approximate solutions we shall derive estimates of derivatives of higher order. By Lemma 3.1. We have known if $\sup_{0 \leq t < -\infty}\|Ag(x,t)\|^2 = 2M$, the approximate solutions satisfy

$$\sup_t \|A^m w_n(t)\| \leq C(M_1)$$

and

$$\sup_t \|A^m u_n(t)\| \le C(M_1)$$

where $m = N/4 - 1/2$.

Lemma 4.1. let $w_n(t)$ be the solution of $(2.1) \to (2.2)$ give above. Suppose that:

$$\sup_t \|g(x,t)\| \le M_2$$

Then we have

$$\sup_t \|\nabla w_n(t)\| \le C(M_1)$$
$$\sup_t \|w_{nt}(t)\| \le C(M_0, M_1, M_2)$$

Proof: by Lemma 3.1. we set m $= \frac{1}{2}$, so there have $\sup_t \|\nabla w_n(t)\| \le \left(2T + \frac{1}{2}\varepsilon^{-1}T\right)\ (c(M_0\ ) + M)$;
from equations $(2.1) \to (2.2)$, we see

$$(u_{nt} + (\alpha\varepsilon - \beta)Au_n - \alpha Au_{nt} - \varepsilon u_n - (1 - \beta\varepsilon + \alpha\varepsilon^2)Aw_n + \varepsilon^2 w_n + \gamma A^2 w_n, u_{nt})$$
$$= (Af(w_n) + g(x,t), u_{nt})\#(4.1)$$

and

$$\|u_{nt}\|^2 + \frac{1}{2}\frac{d}{dt}(\alpha\varepsilon - \beta)\|\nabla u_n\|^2 - \frac{1}{2}\frac{d}{dt}\varepsilon\|u_n\|^2$$
$$\le \alpha\|\nabla u_{nt}\|^2 + (1 - \beta\varepsilon + \alpha\varepsilon^2)\|Aw_n\|\|u_{nt}\| - \varepsilon^2\|w_n\|\|u_{nt}\| - \gamma\|A^2 w_n\|\|u_{nt}\|$$
$$+\|Af(w_n)\|\|u_{nt}\| + \|g(x,t)\|\|u_{nt}\|\#(4.2)$$

where using the Holder inequality.

for the left, by (3.7), we have

$$\frac{1}{2}\frac{d}{dt}(\alpha\varepsilon - \beta)\|\nabla u_n\|^2 - \frac{1}{2}\frac{d}{dt}\varepsilon\|u_n\|^2 \ge \frac{1}{2}\frac{d}{dt}(\alpha\varepsilon - \beta)\mu_1\|u_n\|^2 - \frac{1}{2}\frac{d}{dt}\varepsilon\|u_n\|^2 = [(\alpha\varepsilon - \beta)\mu_1 - \varepsilon]\frac{1}{2}\frac{d}{dt}\|u_n\|^2$$

for the right, we have

$$[\|Af(w_n)\| + \|g(x,t)\| + (1 - \beta\varepsilon + \alpha\varepsilon^2)\|Aw_n\| - \varepsilon^2\|w_n\| - \gamma\|A^2 w_n\|]\|u_{nt}\| \le$$
$$\frac{1}{2}[\|Af(w_n)\| + \|g(x,t)\| + (1 - \beta\varepsilon + \alpha\varepsilon^2)\|Aw_n\| - \varepsilon^2\|w_n\| - \gamma\|A^2 w_n\|]^2 + \frac{1}{2}\|u_{nt}\|^2$$

where using the Young inequality.

for

$$\|Af(w_n)\| = \|f''(w_n)\|\|\nabla w_n\|^2 + \|f'(w_n)\|\|Aw_n\| \le c(M_1)$$

so

$$\frac{1}{2}[\|Af(w_n)\| + \|g(x,t)\| + (1 - \beta\varepsilon + \alpha\varepsilon^2)\|Aw_n\| - \varepsilon^2\|w_n\| - \gamma\|A^2 w_n\|]^2 \le C^2(M_0, M_1)$$

by (3.7), we have

$$\alpha\|\nabla u_{nt}\|^2 + \frac{1}{2}\|u_{nt}\|^2 \le \alpha\|\nabla u_{nt}\|^2 + \frac{1}{2}\mu_2^{-1}\|\nabla u_{nt}\|^2 = \left(\alpha + \frac{1}{2}\mu_2^{-1}\right)\|\nabla u_{nt}\|^2$$

the equation (4.2) ends up being

$$\sup_t \left\|u_{nt(t)}\right\|^2 \le \mathrm{C}^2(M_0, M_1, M_2)\#(4.3)$$

therefor

$$\sup_t \left\|u_{nt(t)}\right\| \le \mathrm{C}(M_0, M_1, M_2)\#(4.4)$$
$$\sup_t \|w_{nt}(t)\| \le \mathrm{C}(M_0, M_1, M_2)\#(4.5)$$

This completes the proof of lemma 4.1.

Lemma 4.2. Let $w_n(t)$ be the approximate solutions give above. Then, we have

$$\sup_t \|Aw_n(t)\| \le C(M_0, M_1, M_2)\#(4.6)$$
$$\sup_t \|\nabla w_{nt}(t)\| \le C(M_0, M_1, M_2)\#(4.7)$$
$$\sup_t \|Aw_{nt}(t)\| \le C(M_0, M_1, M_2)\#(4.8)$$
$$\sup_t \|w_{ntt}(t)\| \le C(M_0, M_1, M_2)\#(4.9)$$

Proof: from (2.1) → (2.2), we can see

$$(u_{nt} + (\alpha\varepsilon - \beta)Au_n - \alpha Au_{nt} - \varepsilon u_n + (\beta\varepsilon - \alpha\varepsilon^2 - 1)Aw_n + \varepsilon^2 w_n + \gamma A^2 w_n, Au_n) = (Af(w_n) + g(x,t), Au_n)$$

and

$$(\alpha\varepsilon - \beta)\|Au_n\|^2 \le \|u_{nt}\|\|Au_n\| + \alpha\|Au_{nt}\|\|Au_n\| + \varepsilon\|u_n\|\|Au_n\| + (\beta\varepsilon - \alpha\varepsilon^2 - 1)\|Aw_n\|\|Au_n\| + \varepsilon^2\|w_n\|\|Au_n\| + \gamma\|A^2 w_n\|\|Au_n\| + \|Af(w_n)\|\|Au_n\| + \|g(x,t)\|\|Au_n\|$$

by the known, we can find

$$\|Aw_n(t)\| \le C(M_0, M_1, M_2)$$

by the equation (3.14) and the definition of u_n, respectively when $m = \frac{1}{2}$ and $m = 1$, we can get

$$\|\nabla w_{nt}(t)\| \le C(M_0, M_1, M_2)$$

and

$$\|Aw_{nt}(t)\| \le C(M_0, M_1, M_2)$$

moreover, similarly, By the equation (4.4) and the definition of u_n, there have

$$\|w_{ntt}(t)\| \le C(M_0, M_1, M_2)$$

Consequently, the proof of Lemma 4.2 is complete.

SUMMARIZE

Firstly, we shall show the convergence of the approximate solutions $w_n(t)$ obtained above. Since the estimates in Lemma 3.1, Lemma 4.1, and Lemma 4.2 are valid, standard compactness arguments imply that there exists a subsequence $w_n(t)$ tending to a function $w(t)$ in such a way

$$w_n \to w \ \ weakly^* \ in \ L_\infty(T; D(A))\#(5.1)$$

$$w_n \to w \ \ strongly \ in \ L_\infty\left(T; D\left(A^{1/2}\right)\right)\#(5.2)$$
$$w_{nt} \to w_t \ weakly^* \ in \ L_\infty\left(T; D\left(A^{1/2}\right)\right)\#(5.3)$$
$$w_{nt} \to w_t \ strongly \ in \ (T; H_\sigma)\#(5.4)$$

where the function $w(t)$ satisfies

$$w \in H^2(T; H_\sigma) \cap H^1(T; D(A)) \cap L_\infty(T; D(A))$$

here, (5.1) → (5.3) are evident, and hence it is sufficient to show the convergence (5.4).

Next, considering unique of the solution for equation (2.1) → (2.2).

Let $(u_i, w_i)(i = 1,2)$ be the solutions of the problem. Set $(x, y) = (u_1 - u_2, w_1 - w_2)$, $f(n) \in C^{k-1}$, $\|f'(w)\| \le Aw^\rho, (\rho > 0)$. Then (x, y) satisfies the equation

$$y_t + (\alpha\varepsilon - \beta)Ay - \alpha Ay_t - \varepsilon y - (1 + \alpha\varepsilon^2 - \beta\varepsilon)Ax + \varepsilon^2 x + \gamma A^2 x - A\left(f(w_1) - f(w_2)\right) = 0\#(5.5)$$
$$y = x_t + \varepsilon x\#(5.6)$$
$$(x, y)(t + T) = (x, y)(t)\#(5.7)$$

taking the inner product of (5.5) with $(-A^{-1})$y, it follows that

$$\frac{1}{2}\frac{d}{dt}\left[(1+\alpha)\|y\|_{H^{-1}(\Omega)} + (1 + \alpha\varepsilon^2 + \varepsilon^2)\|x\|^2\right] + \frac{1}{2}\theta\left[(1+\alpha)\|y\|_{H^{-1}(\Omega)}{}^2 + (1 + \alpha\varepsilon^2 + \varepsilon^2)\|x\|^2\right]$$
$$-(\beta\alpha^{-1}\varepsilon^{-1} + \beta\varepsilon^{-1} + \varepsilon)\|y\|_{H^{-1}(\Omega)} + (f(w_1) - f(w_2), y) \le 0\#(5.8)$$
$$(\theta = 2\varepsilon > 0)$$

since

$$
\begin{aligned}
(f(w_1)-f(w_2),y) &= \left(\int_0^1 f'(w_2+sx)xds\,,y\right) \le \|y\|_{H^{-1}(\Omega)} \left\|D\int_0^1 f'(w_2+sx)xds\right\| \\
&\le \|y\|_{H^{-1}(\Omega)} \left\{\left\|Dx\int_0^1 f'(w_2+sx)ds\right\| + \left\|x\int_0^1 f''(w_2+sx)dsD(w_2+sx)\right\|\right\} \\
&\le \|y\|_{H^{-1}(\Omega)} \{\|Dx\|A(\|w_2\|_{L^\infty(\Omega)}+\|w_1\|_{L^\infty(\Omega)})^\rho + c^*c_f\|Dx\|(\|Dw_1\|+\|Dw_2\|)\} \\
&\le (\beta\alpha^{-1}\varepsilon^{-1}+\beta\varepsilon^{-1}+\varepsilon)\|y\|_{H^{-1}(\Omega)} + \frac{1}{4(\beta\alpha^{-1}\varepsilon^{-1}+\beta\varepsilon^{-1}+\varepsilon)}\|Dx\|^2\left[2^\rho AC^\rho(M_0)+2c^*c_fC(M_1)\right]^2
\end{aligned}
$$

it follows that we obtain

$$(1+\alpha)\|y\|_{H^{-1}(\Omega)} + (1+\alpha\varepsilon^2+\varepsilon^2)\|x\|^2 \le \left((1+\alpha)\|y\|_{H^{-1}(\Omega)} + (1+\alpha\varepsilon^2+\varepsilon^2)\|x\|^2\right)(0)exp(-\theta t), \forall t \ge 0$$

since (x,y) is T-periodic in t, for any $t > 0$, and for any positive integer N, have

$$\left((1+\alpha)\|y\|_{H^{-1}(\Omega)} + (1+\alpha\varepsilon^2+\varepsilon^2)\|x\|^2\right)(t) = \left((1+\alpha)\|y\|_{H^{-1}(\Omega)} + (1+\alpha\varepsilon^2+\varepsilon^2)\|x\|^2\right)(t+NT)$$

so, it follows

$$(1+\alpha)\|y\|_{H^{-1}(\Omega)} + (1+\alpha\varepsilon^2+\varepsilon^2)\|x\|^2 \le \left((1+\alpha)\|y\|_{H^{-1}(\Omega)} + (1+\alpha\varepsilon^2+\varepsilon^2)\|x\|^2\right)(0)exp(-\theta Nt)$$

which implies $(1+\alpha)\|y\|_{H^{-1}(\Omega)} + (1+\alpha\varepsilon^2+\varepsilon^2)\|x\|^2 = 0$. The proof of unique is complete.

REFERENCES

1. Du X. Random attractor of nonlinear strain waves with white noise. Journal of applied analysis and computation. 2011 Jan 1;1(3):361-72.
2. Crauel H, Flandoli F. Attractors for random dynamical systems. Probability Theory and Related Fields. 1994 Sep 1;100(3):365-93.
3. Crauel H, Debussche A, Flandoli F. Random attractors. Journal of Dynamics and Differential Equations. 1997 Apr 1;9(2):307-41.
4. Hisako Kato, Existence of periodic solutions of the Naiver-Stokes equations, J. Math. Anal. Appl. 208 (1997), 141–157.
5. Guo B, Du X, Existence of the periodic solution for the weakly damped Schrodinger-Boussinesq equation. Journal of mathematical analysis and applications. 2001 Oct 15;261(2):453-472.
6. Guo B, Shen L. The global solution of initial value problem for nonlinear Schrödinger-Boussinesq equation in 3-dimensions. Acta Mathematicae Applicatae Sinica. 1990 Jan 1;6(1):11-21.
7. Guo B. The global solution of the system of equations for complex Schrödinger field coupled with Boussinesq type self-consistent field. Acta Math. Sinica. 1983;26(3):295-306.